Krishna's

Numerical Methods

As per U.P. UNIFIED Syllabus (*w.e.f.* 2013–2014)

(*for B.C.A. Vth Sem. Students of all Colleges Affiliated to Universities in Uttar Pradesh*)

By

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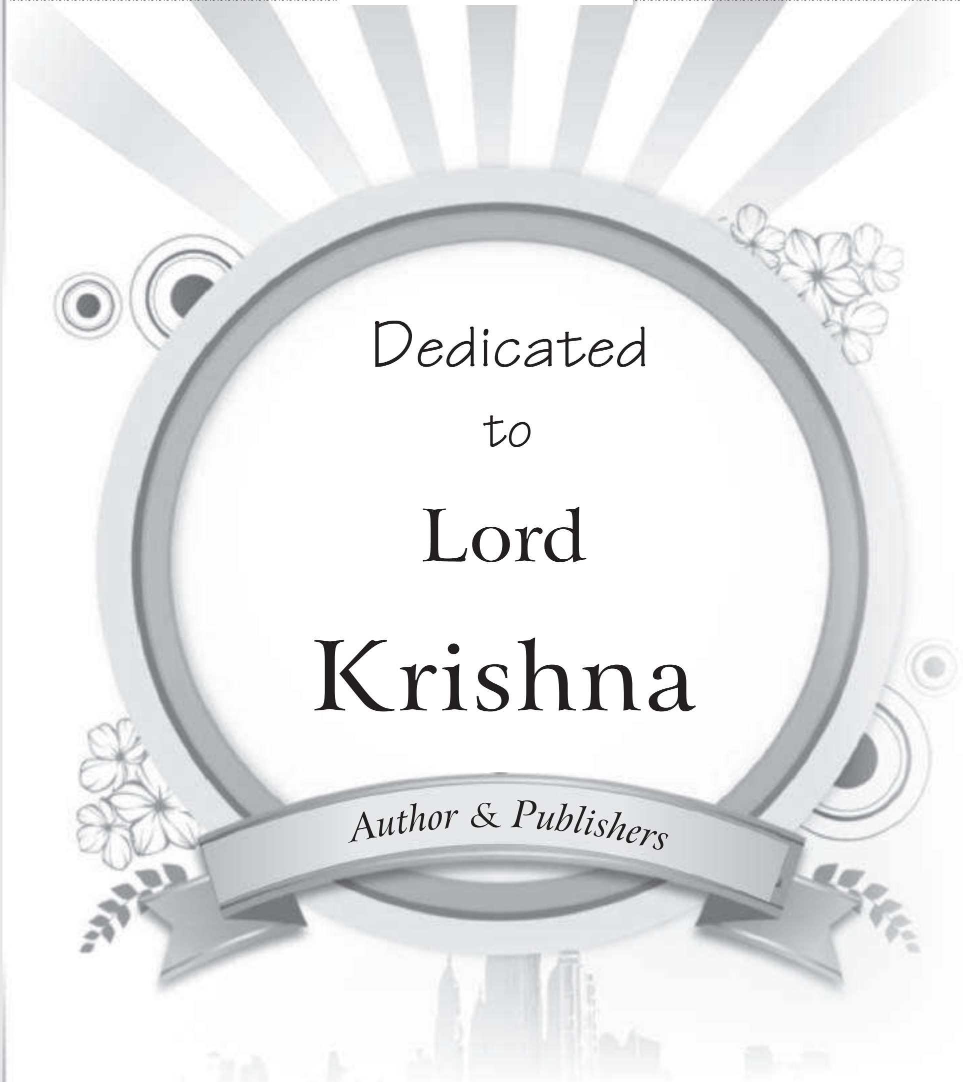

Dedicated
to
Lord
Krishna

Author & Publishers

ABOUT THE BOOK

- The book covers the U.P. UNIFIED syllabus of Vth semester of all U.P. (B.C.A.) colleges.

- The book has been written in a simple, logical and systematic way.

- To make the matter easy for understanding main topics have been divided into sub-topics and illustrative examples with complete solutions are given after theoretical discussion of each topic. When ever possible, alternative methods of solving problems are given.

- The basic concept of each topic is carefully and clearly explained.

- Each chapter includes a problem set containing routine, intermediate and challenging exercises.

- The book has been written to meet the requirements of students pursuing B.C.A. in colleges affiliated to Universities in U.P. .

- Solution of all problem sets are given at the end of each chapter for learners to practice.

- The goal of the book is to present a wide variety of applications of **Numerical Methods** in a flexible and accessible format.

- I have also made extensive use of question papers of the past ten years of **Meerut, Agra, Kanpur, Rohilkhand, Avadh, Bundelkhand, Lucknow, Purvanchal and Other Indian Universities**, so that after studying according to present book, the students will have a clear understanding of the subject and the type of questions they can expect at the university examination.

- Large number of attractive and accurate figures and graphs have been drawn which enable a student to grasp the subject in an easier way.

— J.P. Chauhan

PREFACE

It gives me pleasure to place the book '**Numerical Methods**' (All Universities in U.P. for B.C.A. Vth semester) which has been completely made upto date in light of suggestions received from the students and esteemed teachers of various universities.

The book is so written that it is self-contained for independent study as well. The arrangement and presentation is simple. The chapters have been arranged in a logical order. The theorems and problems have been stated carefully and whenever necessary complete proofs of the theorems have been given. All chapters are arranged according to the latest U.P. Unified Syllabus implemented from 2013-14 keeping in view the requirement of the students of B.C.A. Vth semester of all universities in Uttar Pradesh (U.P.). Errors and omissions have been corrected. Up-to-date year wise references from various universities examination papers have been given throughout the book.

Despite our best endeavours, some errors might have crept into the book. We hope to rectify them in the later reprints of this book through the kind cooperation of the readers.

We hope that the book in its present form will prove to be benificial to the students. We shall also wish to receive your suggestions and comments about the book so that it could be further improved.

— J.P. Chauhan

ACKNOWLEDGEMENT

This book has been written in simple and lucid style for better understanding and appreciation of study. It has been compiled systematically to enable everyone to evaluate detail of the book.

Needless to say, I am deeply indebted to the publisher **Shri Satyendra Kumar Rastogi** *(M.D.),* **Shri Sugam Rastogi** *(E.D.)* and **Mrs. Kanupriya Rastogi** *(Director)* for providing continuous support and enthusiasm.

This acknowledgement would fall short of completion, if I fail to mention the entire team of **Krishna Prakashan Media (P) Ltd.** I would like to thank the contributors for their excellent work and especially for their creativity .

Finally, we shall be very grateful to all instructors and students who send us their helpful suggestions.

—J.P. Chauhan

SYLLABUS

NUMERICAL METHODS

U.P. UNIFIED (*w.e.f.* 2013-2014)

B.C.A. – S 304	**B.C.A. Vth Semester**	**L T P C**
		3 1 0 4

UNIT-1

Roots of Equations: Bisections Method, False Position Method, Newton's Raphson Method, Rate of convergence of Newton's method.

UNIT-2

Interpolation and Extrapolation: Finite Differences, The operator E, Newton's Forward and Backward Differences, Newton's divided differences formulae, Lagrange's Interpolation formula for unequal Intervals, Gauss's Interpolation formula, Starling formula, Bessel's formula, Laplace- Everett formula.

UNIT-3

Numerical Differentiation Numerical Integration: Introduction, direct methods, maxima and minima of a tabulated function, General Quadratic formula, Trapezoidal rule, Simpson's One third rule, Simpson's three- eight rule.

UNIT-4

Solution of Linear Equation: Gauss's Elimination method and Gauss's Siedel iterative method.

UNIT-5

Solution of Differential Equations: Euler's method, Picard's method, Fourth-order Runge-Kutta method.

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Chapter 3: Interpolation with Equal Intervals .... (59-76)
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Chapter 5: Central Difference Interpolation Formulae .... (105-132)

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Chapter 7: Numerical Integration .... (161-186)

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❍❍❍

Unit

1

Chapter-1: Roots of Equations

Unit-1

Chapter 1

Roots of Equations

1.1 Method for the Finding the Roots (Zeros) of an Equation

[B.C.A. (Agra) 2007, 2008]

We shall now discuss few methods which are applicable to both algebraic (with numerical coefficients) and transcendental equations. In finding the real roots of an equation, we first find an approximate value of the root of the given equation and then successively improve it to some desired degree of accuracy. The general technique is that we start with an initial approximate value, say x_0 and then find the better approximations successively $x_1, x_2, \ldots, x_n$ by repeating the same method. If the successive approximations at each step of a method approach the root more and more closely, we say that the method *converges.*

1.2 Bisection Method (Bolzano Method)

[B.C.A. (Rohilkhand) 2009; B.C.A. (Kanpur) 2006]

If a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are the opposite signs then there exists at least one root between a and b. Let $f(x_0)$ be negative and $f(b)$ be positive so that the approximate value of the root between them is $x_0 = (a+b)/2$. If $f(x_0) = 0$, then it asserts that x_0 is the correct root of $f(x) = 0$. On the other hand, if $f(x_0) \neq 0$, then the root

a– – – • – – –b

either lies in between $\left(a, \frac{a+b}{2}\right)$ or $\left(\frac{a+b}{2}, b\right)$

depending on wether $f(x_0)$ is negative or positive. We again bisect the interval and repeat the process until the root is obtained to desired accuracy.

SOLVED EXAMPLES

Example 1: ***To find a real root of the equation.***

$$f(x) = x^3 - x - 1 = 0$$

[B.C.A. (Lucknow) 2008]

Solution: Here $f(1) = 1-1-1 = -1$, $f(2) = 8-2-1 = 5$; hence the root lies in between (1, 2) *i.e.,* the first approximation is $x_0 = (3/2)$, again $f(3/2) = (7/8)$ *i.e.,* positive hence the root is in between (1, 3/2); the second approximation being $x_1 = \frac{1+1.5}{2} = 1.25$.

Repeating this process, we get the successive approximations as

$$x_2 = \frac{1.25+1.5}{2} = 1.375,\ x_3 = 1.3125$$

$$x_4 = 1.34375 \text{ etc.}$$

1.3 Regula Falsi Method: (Method of False Position)

[B.C.A. (Agra) 2003, 2004, 2007; B.C.A. (Avadh) 2004]

This is the most oldest method for finding the real root of a numerical equation $f(x) = 0$. To find the real root of the equation $f(x) = 0$, we consider a sufficiently small interval (x_1, x_2) in which the root lies. Since the root lies in the interval (x_1, x_2), the curve must cross the axis of x in between (x_1, x_2) and hence $y_1 = f(x_1)$, $y_2 = f(x_2)$ have opposite signs.

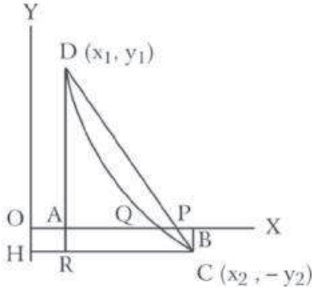


Fig. 1.1

This method is based on the principal that any portion of a smooth curve is practically straight for a short distance and hence we may say that the x-co-ordinate of the point of intersection of the straight line joining $(x_1, y_1), (x_2, y_2)$ and the axis of x gives an approximate value of the desired root. By the figure it is obvious that $x = OP$ is the approximate value of the desired root whereas $x = OQ$ is the actual value of the root. Now we shall derive a formula to compute the root.

We have $$OP = OA + AP = x_1 + AP \qquad \text{...(1)}$$

The triangles *DPA* and *DRC* are similar, so we have

$$\frac{DA}{AP} = \frac{DR}{CR} \Rightarrow AP = \frac{CR}{DR}\,DA = \frac{(x-x_1)\,DA}{DA+AR} = \frac{(x_2-x_1)\left|y_1\right|}{\left|y_1\right|+\left|y_2\right|}$$

$$\therefore \qquad OP - OA = \frac{\left|x_2-x_1\right|\,\left|y_1\right|}{\left|y_1\right|+\left|y_2\right|} \Rightarrow OP = x_1 + \frac{\left|x_2-x_1\right|\,\left|y_1\right|}{\left|y_1\right|+\left|y_2\right|}$$

$$= \text{Desired value of the root.}$$

Let the desired value (which is approximate) be represented by $x^{(1)}$; then to find the better approximation, we evaluate $y^{(1)}$ by $y^{(1)} = f(x)^{(1)}$. Now either $y^{(1)}$ and y_1 or y_2 will be of opposite signs. In case $y^{(1)}$ and y_1 are of opposite signs then one root will lie the interval $(x_1, x^{(1)})$. Now again apply the method of false position to this interval to get second approximation. In case $y^{(1)}$ and y_2 are of opposite signs then second approximation is obtained by using the method of false position to the interval $(x^{(1)}, x_2)$.

Continuing this process we can obtain the root to the desired degree of accuracy.

1.4 Newton-Raphson Method

[B.C.A. (Meerut) 2012; B.C.A. (Agra) 2003, 2004, 2008, 2009; B.C.A. (Kanpur) 2004, 2010; B.C.A. (Lucknow) 2006, 2009, 2012]

Let x_0 be an approximate value of a root of the equation $f(x) = 0$, and let $x_0 + h$ be the exact value of the corresponding root, where h is very small quantity.

Then $$f(x_0 + h) = 0 \qquad \text{...(1)}$$

Since $x_0 + h$ is the root of the equation $f(x) = 0$.

Expanding (1) by Taylor's theorem, we get

$$f(x_0 + h) = f'(x_0) + hf'(x_0) + \frac{h^2}{2\,!} f''(x_0) + \ldots = 0$$

Since h is very small, neglecting second and higher order terms and taking the first approximation, we have

$$f(x_0) + hf'(x_0) = 0 \quad \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}, \text{ provided } f'(x_0) \neq 0.$$

$$\therefore \qquad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}. \qquad \text{...(2)}$$

Relation (2) gives the improved value of the root over the previous one. Now substituting x_1 for x_0 and x_2 for x_1 in (2), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{...(3)}$$

In general, we can get an approximation or the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{...(4)}$$

From this formula, we can calculate successive better values of the root. Formula (4) is known as ***Newton-Raphson Method***.

From above it is obvious that this method is applicable only when h is very small *i.e.,* when $f'(x)$ is large, in other words when the graph of the function near the root is nearly vertical.

If $f(x)$ is small in the neighbourhood of the root then the value of h is large and so by this method the computation of the root will be slow process or may fail altogether. Therefore, this method is not suitable in those cases where graph of $f(x)$ is nearly horizontal where it crosses the x-axis. This method can be used for both the algebraic and transcendental equations and also works for complex coefficients of an equation.

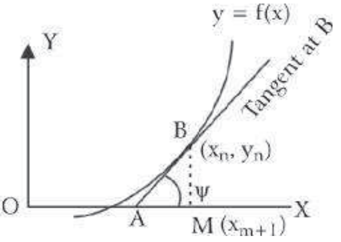


Fig. 1.2

Geometrical interpretation of Newton-Raphson's Method.

Equation of the curve is $y = f(x)$

$$\Rightarrow \qquad (dy/dx) = f'(x),$$

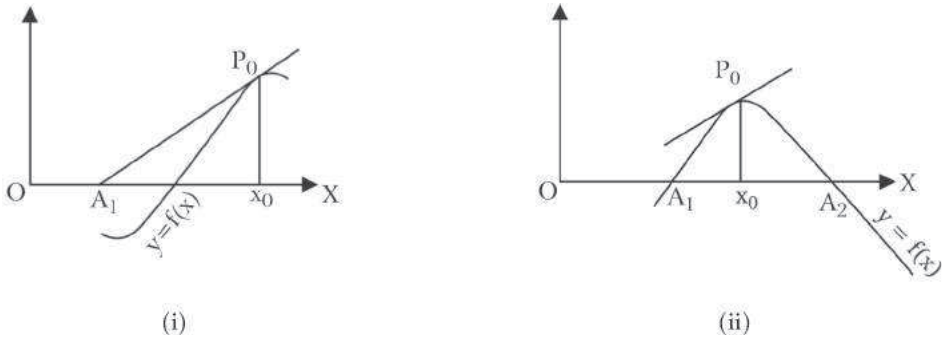
$$\Rightarrow \qquad \text{Slope of the tangent at B} = f'(x_n). \qquad \ldots(5)$$

At the point A, $y = 0$ *i.e.* $f(x) = 0$ or in other words, the point A is closer to the desired root than B. Let the co-ordinates of the point be $(x_{n+1}, 0)$. Then we have

$$f'(x_n) = \frac{BM}{AM} = \frac{f(x_n)}{OM - OA} = \frac{f(x_n)}{x_n - x_{n+1}} \text{ implying } x_{n+1} - x_n = \frac{-f(x_n)}{f'(x_n)} \qquad \ldots(6)$$

1.4.1 Cases of Failure of Newton's Method **[B.C.A. (Rohilkhand) 2009, 2011]**

As discuss earlier, Newton's method fails if $f(x_0)/f'(x_0)$ is not small enough. This happens, when the graph of the function $y = f(x)$ near the root is flat. Sometimes, the curve is not rather flat but a wrong choice of initial approximation leads to the failure of Newton's method.



(i) (ii)

Fig. 1.3

The tangent at P_0 $(x = x_0)$ meets the x-axis at A_1 thus giving more worse approximation of the root. This situation can be avoided by choosing initial approximation more close to P_0.

Another case of failure is the situation where two roots of the equation are close together. In this case, the simplest initial approximation x_0 is also nearly parallel to x-axis.

1.4.2 Comparison of Newton's Method with the Regula-Falsi Method

[B.C.A. (Agra) 2005, 2011]

Newton's method is conditionally convergent while Regula-Falsi is surely convergent. Once Newton's method converges, it converges faster. In Regula-Falsi method we calculate only one more value of function at each step *i.e.* $f(x)^{(n)}$, while Newton's method requires calculations of $f(x_n)$ and $f'(x_n)$ at each step. Hence Newton's method generally requires less number of iterations but requires more time for computations at each iteration. Newton's method is not always the best method a given problem, but due to its formal simplicity and faster speed, people often use it as the first method to solve the problem.

Again when $f'(x)$ is large near the root, the correction to be applied is smaller in case of Newton's method and then this method is preferred while if $f'(x)$ is small near the root then correction to be applied is large and the curve becomes parallel to x-axis, so in this case Regula-falsi method must be applied.

Generalised Newton's Method for Multiple Roots: If α is a root of $f(x)=0$ which is repeated m times, then we have

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}$$

This is called *generalised Newton's formula.* It reduces to Newton-Raphson formula at $m=1$.

Note 1: If α is a root of $f(x)=0$ with multiplicity m, then it is also a root of $f'(x)=0$ with multiplicity $m-1$, or $f''(x)=0$ with multiplicity $m-2$ and so on. Hence, if initial approximation x_0 is sufficiently close to the root, then the expressions.

$$x_0 - m\,\frac{f(x_0)}{f'(x_0)},\; x_0 - (m-1)\,\frac{f'(x_0)}{f''(x_0)},\; x_0 - (m-2)\,\frac{f''(x_0)}{f'''(x_0)},\ldots$$

will have the same value.

Note 2: Generalised Newton's formula has a second order of convergence for determining a multiple root.

***Example 2:* *Find the double root of the equation* $x^3 - x^2 - x + 1 = 0$.**

[B.C.A. (Rohilkhand) 2008]

Solution: Let $f(x) = x^3 - x^2 + 1 - x$

$\Rightarrow \qquad f'(x) = 3x^2 - 2x - 1,\; f''(x) = 6x - 2$

Starting with $x_0 = 0.9$, we obtain by $x_0 - m\,\dfrac{f(x_0)}{f'(x_0)}, m = 2$

$$x_0 - 2\,\frac{f(x_0)}{f'(x_0)} = 0.9 - \frac{2\times 0.019}{-0.37} = 1.003$$

and $$x_0 - (2-1)\,\frac{f'(x_0)}{f''(x_0)} = 0.9 - \frac{(-0.37)}{3.4} = 1.009$$

Then the closeness of these values simply indicates that there is a double root at $x = 1$. Further taking $x_1 = 1.01$ for the next approximation

we have $$x_1 - 2\,\frac{f(x_1)}{f'(x_1)} = 1.01 - \frac{2 \times .0002}{.0403} = 1.0001$$

$$x_1 - (2-1)\,\frac{f'(x_1)}{f'(x_1)} = 1.01 - \frac{0.0403}{4.06} = 1.0001$$ which means that ∃ a double root

at $x = 1.0001$ close to $x = 1$.

Example 3: *Obtain the cube root of* 12.

[B.C.A. (Meerut) 2002, 2004; B.C.A. (Kurukshetra) 2005, 2008, 2013]

Solution: Let $f(x) = N - x^3, \ \Rightarrow f'(x) = -3x^2$.

$$\Rightarrow \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - = x_n - \frac{N - x_n^3}{-3x_n^2} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

(*General formula of cube root of N*)

Now we shall obtain cube root of 12.

We have $(8)^{1/3} = 2$ and $(27)^{1/3} = 3$; talking $x_0 = 2.5$, we obtain

$$x_1 = \frac{1}{3}\left[2 \times 2.5 + \frac{12}{(2.5)^2}\right] = \frac{1}{3}\left[5 + \frac{48}{25}\right] = 2.3066$$

$$x_2 = \frac{1}{3}\left[2 \times 2.306 + \frac{12}{(2.3006)^2}\right] = 2.2901$$

Similarly, higher approximations can be obtained.

1.4.3 Rate of Convergence of Newton's Method

[B.C.A. (Meerut) 2012; B.C.A. (Agra) 2004; B.C.A. (Kanpur) 2007, 2010, 2013]

Let p be the exact value of the root and let ε_n be the small quantity by which x_n differs from b *i.e.* $x_n = b + \varepsilon_n$, implying $x_{n+1} = b + \varepsilon_{n+1}$, whence Newton-Raphson's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad i.e. \ \ b + \varepsilon_{n+1} = b + \varepsilon_n - \frac{f(b+\varepsilon_n)}{f'(b+\varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(b+\varepsilon_n)}{f'(b+\varepsilon_n)}\,b \qquad \text{...(1)}$$

By Taylor's theorem, we have

$$f(b+\varepsilon_n) = f(b) + \varepsilon_n f'(b) + \frac{\varepsilon_n^2}{2!} f''(b)$$

$$= \varepsilon_n f'(b) + \frac{\varepsilon_n^2}{2!} f''(b) + \ldots \qquad (\because \ f(b) = 0)$$

and $f'(b+\varepsilon_n) = f'(b) + \varepsilon_n f''(b) + \ldots$ Putting these values in (1), we obtain

$$\varepsilon_{n+1} = \varepsilon_n - \varepsilon_n\,\frac{f'(b) + \frac{\varepsilon_n}{2!} f''(b) + \ldots}{f'(b) + \varepsilon_n f''(b) + \ldots}$$

$$= \frac{\varepsilon_n^2 f''(b)}{2\,[f'(b) + \varepsilon_n f''(b)]} = \varepsilon_n^2 \frac{f''(b)}{2f'(b)} \frac{1}{\left[1 + \varepsilon_n \frac{f''(b)}{2f'(b)}\right]}$$

$$= \frac{\varepsilon_n^2}{2f'(b)} f''(b) \left\{1 + \varepsilon_n \frac{f''(b)}{f'(b)}\right\}^{-1} \approx \varepsilon_n^2 \frac{f''(b)}{2f'(b)} \qquad \text{...(2)}$$

From (2) it is obvious that the subsequent error is proportional to the square of the previous error and hence we may say that the convergence is quadratic.

1.5 Nearly Equal Roots

We know that Newton's Method is applicable only when $f(x) \neq 0$ in the neighbourhood of the actual root $x = b$ *i.e.* in the interval $(b - h, b + h)$. But in practice the above restriction may not be satisfied and further, the quantity h which satisfies this may be very small. This situation arises when the roots are very close to each other. To avoid this difficulty, we proceed as follows:

By Rolle's Theorem, we know that $f'(x) = 0$ vanishes at least one time between two roots of $f(x) = 0$ and hence we can choose h sufficiently small to satisfy the interval $(b - h, b + h)$ in case of nearly equal roots. When two roots are equal, say $x = b$, then $f(x)$ and $f'(x)$ both vanish at $x = b$. Hence while carrying Newton's method if x_i is very nearly the roots of $f(x)$ and $f'(x)$ *i.e.* $f(x_j)$ and $f'(x_j)$ both are very small, then we shall proceed to obtain two new starting values for the two nearly equal roots. In order to obtain these two values for the two nearly equal roots. In order to obtain these two values we shall first apply Newton's method to the equation $f'(x) = 0$ *i.e.*, we use iteration formula

$$x_{j+1} = x_j - \frac{f'(x_j)}{f''(x_j)} \qquad \text{...(1)}$$

*Let $x = d$ be the solution obtained by (1), then applying Taylor's Theorem, we get

$$f(x) = (d + (x - d))$$

$$= f(d) + (x - d)\, f'(d) + \frac{(x - d)^2}{2\,!} f''(d) + \ldots$$

$$\Rightarrow \qquad f(x) = f(d) + \frac{1}{2\,!}(x - d)^2 f''(d) + \ldots \qquad [\because\ f'(d) = 0]$$

Leaving cube and higher powers of $(x - d)$, we get

$$f(x) = f(d) + \frac{(x - d)^2}{2\,!} f''(d) \qquad \text{...(2)}$$

Hence the zeroes of $f(x)$ near $x = d$ are approximately given by

$$f(x) = 0 \qquad \Rightarrow \quad f(d) + \frac{1}{2\,!}(x - d)^2 f''(d) = 0$$

$$\Rightarrow \qquad x = d \pm \sqrt{\{-2\, f(d) / f''(d)\}}\,. \qquad \ldots(3)$$

Now assuming these values as the starting values, we can find the two close roots by Newton's method.

1.6 Rate of Convergence of Newton's Method When there Exist Double Roots

Let b be the exact value of the root and x_n differs from b by the small quantity ε_n. Then $x_n = b + \varepsilon_n \Rightarrow x_{n+1} = b + \varepsilon_{n+1}$ By Newton-Raphson Method we have $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

or
$$b + \varepsilon_{n+1} = b + \varepsilon_n - \frac{f(b+\varepsilon_n)}{f'(b+\varepsilon_n)}$$

or
$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(b+\varepsilon_n)}{f'(b+\varepsilon_n)}$$

$$= \varepsilon_n - \frac{f(b) + \varepsilon_n f'(b) + \dfrac{\varepsilon_n^2}{2!} f''(b) + \dfrac{\varepsilon_n^3}{3!} f'''(b) + \ldots}{f'(b) + \varepsilon_n f''(b) + \dfrac{\varepsilon_n^2}{2!} f'''(b) + \dfrac{\varepsilon_n^3}{3!} f^{(iv)}(b) + \ldots}$$

(By Tailor's Theorem)

$$= \varepsilon_n - \frac{\dfrac{\varepsilon_n^2}{2!} f''(b) + \dfrac{\varepsilon_n^3}{3!} f'''(b) + \ldots}{\varepsilon_n f''(b) + \dfrac{\varepsilon_n^2}{2!} f'''(b) + \ldots}$$

$[\because \ f(b) = 0 = f'(b)$ as there exist double roots$]$

$$= \varepsilon_n - \varepsilon_n \frac{\dfrac{1}{2} f''(b) + \dfrac{\varepsilon_n}{6} f'''(b)}{f''(b) + \dfrac{\varepsilon_n}{2} f'''(b)} \qquad \text{[leaving higher powers]}$$

$$= \varepsilon_n \frac{\dfrac{1}{2} f''(b) + \dfrac{\varepsilon_n}{3} f'''(b)}{f''(b) + \dfrac{\varepsilon_n}{2} f'''(b)}$$

$$= \varepsilon_n \cdot \frac{1}{2} \frac{f''(b)}{f''(b)} = \frac{1}{2} \varepsilon_n = \frac{2-1}{2} \varepsilon_n,$$

which shows that the convergence is linear. Proceeding in the same manner we can prove that if $x = b$ is root of multiplicity $m > 1$, then the speed of convergence is given by $\varepsilon_{n+1} \approx \dfrac{m-1}{m} \varepsilon_n$, which is also linear.

Example 4: ***The equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between 1.4 and 1.5. Find the root to four places of decimal by false position method.***

[B.C.A. (Bundelkhand) 2010]

Solution: For the first approximation, we take

$$\left.\begin{aligned} x_1 &= 1.4, \; y_1 = -.056061 \\ x_2 &= 1.5, \; y_2 = 1.953125 \end{aligned}\right\}$$

$$\therefore \qquad x^{(1)} = x_1 + \frac{|(x_2 - x_1)| . |y_1|}{|y_2| + |y_1|} = 1.4 + \frac{(.1)\,(.056061)}{1.953125 + 0.056061}$$

$$= 1.4 + .003 = 1.403.$$

For the second approximation, we have

$$\left.\begin{aligned} x_1 &= 1.403, \; y_1 = -.0167 \\ x_2 &= 1.402, \; y_2 = 1.0056 \end{aligned}\right\} \Rightarrow x^{(2)} = 1.403 + \frac{(.001)\,(.0167)}{(.0056) + (.0167)}$$

$$= 1.4037.$$

Again, for the third approximation, we have

$$\left.\begin{aligned} x_1 &= 1.4036, \; y_1 = -.0001 \\ x_2 &= 1.4037, \; y_2 = 1.0004 \end{aligned}\right\} \Rightarrow x^{(3)} = 1.4036 + \frac{(.0001)\,(.0001)}{(.0004) + (.0001)}$$

$$= 1.4036$$

$\Rightarrow$ that the root is 1.4036.

Example 5: ***Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.***

[B.C.A. (Kanpur) 2010; B.C.A. (Avadh) 2011; B.C.A. (Meerut) 2009]

Solution: Let $f(x) = x^3 - 2x - 5$. Then, we have

$$f(2) = -1 \quad \text{and} \quad f(3) = 16.$$

$\Rightarrow$ a root lies between 2 and 3. For the first approximation, we have

$$x_1 = 2, x_2 = 3, y_1 = f(x_1) = -1, f(x_2) = 16 = y_2.$$

Then, we have

$$x^{(1)} = x_1 + \frac{|(x_2 - x_1)| \;.\; |y_1|}{|y_2| + |y_1|} = 2 + \frac{1}{17} = 2.0588.$$

Hence root lies between 2.0588 and 3. Hence $f(2.0588) = -0.3908$

For the second approximation, we again have

$$x_1 = 2.0588, x_2 = 3, f(x_1) = -0.3908, f(x_2) = 16.$$

$$\Rightarrow \qquad x^{(2)} = x_1 + \frac{|(x_2 - x_1)| \;.\; |y_1|}{|y_2| + |y_1|} = 2.0588 + \frac{0.9412}{16.3908}(0.3908)$$

$$= 2.0813$$

⇒ The root lies between 2.0813 and 3.

Repeating this process, the successive approximations are given as

$$x^{(3)} = 2.0862, x^{(4)} = 2.0915, x^{(5)} = 2.0934,$$

$$x^{(6)} = 2.0941, x^{(7)} = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 6: ***Find the root of the equation*** $xe^x = \cos x$ ***using the Regula-Falsi Method correct to four decimal places.*** **[B.C.A. (Rohtak) 2013; B.C.A. (A.M.U.) 2011]**

Solution: Let $f(x) = \cos x - xe^x = 0$

$$\Rightarrow \qquad f(0) = 1, f(1) = \cos 1 - e = -2.17798$$

⇒ the root lies between 0 and 1. Let us take

$$x_1 = 0, x_2 = 1, f(x_1) = y_1 = 1$$

$$\therefore \qquad f(x_2) = -2.17798 = y_2.$$

By Regula-Falsi Method, we then obtain

$$x = x_1 + \frac{|(x_2 - x_1)|.|y_1|}{|y_2| + |y_1|} = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \qquad ...(1)$$

Hence, the root lies in between $(0.31467, 1)$.

Now $\qquad f(0.31467) = 0.51987$

Then taking $x_1 = 0.31467, x_2 = 1, f(x_1) = 0.51987,$

$f(x_2) = -2.17798$ in (1), we get

$$x^{(2)} = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

i.e., the root lie between 0.44673 and 1.

Then $\qquad f(0.4673) = 0.20356$. Then taking $x_1 = 0.44673, x_2 = 1$

$f(x_1) = 0.20356$ and $f(x_2) = -2.17798$ in (1), we obtain

$$x^{(3)} = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are given as

$$x^{(4)} = 0.50995, \; x^{(5)} = 0.51520, \; x^{(6)} = 0.51692$$

$$x^{(7)} = 0.51748, \; x^{(8)} = 0.51767, \; x^{(9)} = 0.51775 \text{ etc.}$$

Thus the root is 0.5177 correct to 4 decimal places.

Example 7: ***Find the real root of the equation $x\log_{10} x - 1.2 = 0$, correct to five places of decimal.*** **[B.C.A. (Agra) 2006, 2013; B.C.A. (Lucknow) 2011]**

Solution: Let $y = f(x) = x\log_{10} x - 1.2$, when we have

$$f(2) = 2\log_{10} 2 - 1.2 = 2\times(.30103) - 1.2 = -.59794$$
$$f(3) = 3\log_{10} 3 - 1.2 = 3\times(.44712) - 1.2 = -.23138$$

Hence, there exists at least one real root of the equation $f(x)=0$ in the interval $(2, 3)$ and it is closer to 3.

For the first approximation, we get

$$\begin{cases} x_1 = 2, \; y_1 = -.59794 \\ x_2 = 3, \; y_2 = .23136 \end{cases} \Rightarrow x^{(1)} = x_1 + \frac{|(x_2 - x_1)|\,|y_1|}{|y_1| + |y_2|} = 2 + \frac{.59704}{.82990} = 2.719$$

Also, we have

$$f(2.721) = 2.719\times\log_{10}(2.719) - (1.2)$$
$$= 2.719\times(.43473) - 1.2$$
$$= 1.1829 - 1.2 = -.0188$$

Further $f(2.721)$ and $f(3)$ are the opposite signs; so the root lies in between $x = 2.721$ and $x = 3.00$. For the second approximation, we take

$$\begin{cases} x_1 = 2.719, \; y_1 = -.3171 \\ x_2 = 3, \; y_2 = .23136 \end{cases}$$

$$\Rightarrow \qquad x^{(2)} = 2.719 + \frac{(3 - 2.719)(.3171)}{(.3171) + (.23136)} = 2.88$$

Also, we have $f(2.702) = 2.7402\times\log_{10}(2.7402) - 1.2$

$$= 2.740 + (0.43778) - 1.2$$
$$= 1.1996 - 1.2 = -.0004$$

Again as $f(2.740200)$ and $f(3)$ are of opposite sign, so that root lies in the interval $(2.7402, 3)$ and it is closer to $x = 2.7402$.

For the third approximation, we have

$$\begin{cases} x_1 = 2.7402, \; y_1 = .0004 \\ x_2 = 3, \; y_2 = .23136 \end{cases} \Rightarrow x^{(3)} = 2.7402 + \frac{(3 - 2.7402)(.0004)}{.0004 + .23136}$$

$$= 2.7385 + .00044844 = 2.7405967$$

$\Rightarrow$ The root is 2.7406.

Example 8: ***Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.***

[B.C.A. (Purvanchal) 2005, 2008; B.C.A. (Rohilkhand) 2009, 2013]

Solution: $f(1) = 1 - 1 - 1 = -1$; $f(2) = 2^3 - 2 - 1 = 5$. Now $f(1)$ is negative and $f(2)$ is positive so at least one root lies between 1 and 2.

So we take $x_0 = \frac{1}{2}(1+2) = (3/2)$.

Again $f(x_0) = (27/8) - (3/2) - 1 = (7/8)$, which is positive. But $f(1)$ and $f(1.5)$ are of opposite signs so the root lies between 1 and 1.5 and therefore, we obtain

$$x_1 = \frac{1+1.5}{2} = 1.25$$

Further, $f(x_1) = -19/64$, which is negative. Hence we conclude that the root lies between 1.25 and 1.5 implying

$$x_2 = \frac{1.25+1.5}{2} = 1.375.$$

Whence repeating the procedure, we obtain the successive approximations as

$$x_3 = 1.3125, x_4 = 1.34375, x_5 = 1.328125 \text{ etc.}$$

Example 9: ***Find the real root of the equation $x^2 + 4\sin x = 0$ correct to four places of decimal by using Newton's-Raphson Method.*** **[B.C.A. (A.M.U.) 2007]**

Solution: We have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$

Let us take $f(x) = x^2 + 4\sin x$,

$$\Rightarrow \qquad f'(x) = 2x + 4\cos x.$$

$$\Rightarrow \qquad x_{n+1} = x_n - \frac{x_n^2 + 4\sin x_n}{2x_n + 4\cos x_n}$$

By the equation it is obvious that it is satisfied only by a negative value of x as x^2 is to be positive always for all real values of x. By the graphical method, approximate value of the root is given by $x_2 = -1.9$.

Again starting from $x_0 = -1.9$, relation (1) gives

$$x_1 = x_0 - \frac{x_0^2 + 4\sin x_0}{2x_0 + 4\cos x_0}$$

$$= -1.9 - \frac{(-1.9)^2 + 4\sin(-1.9)}{2(-1.9) + 4\cos(-1.9)} = 1.9 - \frac{3.61 - 4(0.945)}{-3.8 + (0.323)}$$

$$= -1.93$$

The second approximation,

$$x_1 - \frac{x_1^2 + 4\sin x_1}{2x_1 + 4\cos x_1} = -1.93 - \frac{(-1.93)^2 + 4\sin(-1.93)}{2(-1.93 + 4\cos(-1.93))} = -1.93 - 0.0038$$

$$= 1.9338$$

Example 10: ***Obtain $\sqrt{(12)}$, to five places of decimals by Newton's Raphson Method.***

[B.C.A. (Agra) 2004, 2007, 2009; B.C.A. (Meerut) 2002, 2003, 2006, 2010]

Solution: Let us take $x = \sqrt{(12)}$, so that $x^2 - 12 = 0$. From this it is obvious that $x^2 - 12$ becomes negative for $x = 3$, whereas it is positive for $x = 3.5$. Hence the root of $x^2 - 12 = 0$ lies in the interval $(3, 3.5)$. Further $f(x) = x^2 - 12$, so that the given equation is $f(x) = 0$. Again $f'(x) = 2x$.

Now by Newton's-Raphson Method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \qquad x_1 = x_2 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \frac{f(3.5)}{f'(3.5)} \qquad [\text{Taking } x_0 = 3.5]$$

$$= 3.5 - \frac{(3.5)^2 - 12}{2\,(3.5)} = 3.464.$$

Again, we have

$$x_2 = x_1 - \frac{x_1^2 - 12}{2\,x_1} = 3.464 - \frac{(3.464)^2 - 12}{2\,(3.464)} = 3.4641$$

$$x_3 = x_2 - \frac{x_2^2 - 12}{2\,x_2} = 3.4641 - \frac{(3.4641)^2 - 12}{2\,(3.4641)} = 3.4641$$

Hence up to four places of decimals, we get

$$\sqrt{(12)} = 3.4641.$$

Example 11: ***The equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between (1.4, 1.5). Find this root to four decimal places by False Position Method.***

[B.C.A. (Agra) 2007; B.C.A. (Kanpur) 2008]

Solution: For the first approximation, we take

$$\begin{cases} x_1 = 1.4, \; y_1 = -0.56064 \\ x_2 = 1.5, \; y_2 = 1.953125 \end{cases}$$

$$\Rightarrow \qquad x^{(1)} = x_1 + \frac{|x_2 - x_1|\,|y_1|}{|y_1| + |y_2|} = 1.4 + \frac{.0056064}{2.009189} = 1.403.$$

For the second approximation, we have

$$\begin{cases} x_1 = 1.403, \; y_1 = -.0167 \\ x_2 = 1.404, \; y_2 = .0056 \end{cases}$$

$$\Rightarrow \qquad x^{(2)} = 1.403 + \frac{(.001)\,(.0167)}{(.0056) + (.0167)} = 1.403 + .0007 = 1.4037$$

For third approximation, we have

$$\begin{cases} x_1 = 1.4036, \; y_1 = -.0001 \\ x_2 = 1.4037, \; y_2 = .0004 \end{cases}$$

$$\Rightarrow \qquad x^{(2)} = 1.4036 + \frac{(.0001)\,(.0001)}{(.0001) + (.0004)} = 1.4036 + 0.00002 = 1.4036$$

Hence the root is 1.4036

Example 12: ***Find the approximate value for the real root of $x \log_{10} x - 1.2 = 0$ correct to five decimal places by Newton-Raphson Method.*** **[B.C.A. (Lucknow) 2006]**

Solution: We have $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$; $n = 0, 1, 2$ ....

Here, $f(x) = x \log_{10} x - 1.2$

$\Rightarrow$ $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \cdot \log_{10} e = \log_{10} x + .43429$

$\therefore$ $$x_{n+1} = x_n - \frac{x_n \log_{10}(x_n) - 1.2}{\log_{10} x_n + .43429} = \frac{.43429\, x_n + 1.2}{\log_{10} x_n + .43429}.$$

Now, putting $n = 0$, we get

$$x_1 = \frac{.43429\,(2) + 1.2}{\log_{10} 2 + .43429} = 2.81 \qquad [\text{starting with } x_0 = 2]$$

Similarly, $$x_2 = \frac{.13429\,(2.81 + 1.2)}{\log_{10} 2.81 + .43429} = 2.741$$

$$x_3 = \frac{.43429\,(2.741) + 1.2}{\log_{10} 2.741 + .43429} = 2.74064$$

$$x_4 = \frac{.43429\,(2.74064) + 1.2}{\log_{10} 2.74064 + .43429} = 2.74065$$

$$x_5 = \frac{.43429\,(2.74065) + 1.2}{\log_{10} 2.74065 + .43429} = 2.74065$$

Hence, the required root is 2.74065.

Example 13:* *Apply Newton's method to obtain the root of the equation $x \log_{10} x = 4.7772393$, correct to six decimal places.

Solution: By the logarithmic table, we have $6 \log_{10} 6 = 4.64$ and $7 \log_{10} 7 = 5.92$; so we take the first approximation as $x_0 = 6$. Now applying Newton's method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \Rightarrow \quad x_1 = 6 - \frac{5 \times 0.77815 - 4.7772393}{\log_{10} 6 + \log_{10} e}$$

$\Rightarrow$ $$x_1 = 6 + \frac{0.1083}{0.778 + 0.434} = 6 + \frac{0.1083}{1.212} = 6.089.$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x)}{f'(x_1)} \Rightarrow x_2 = 6.89 - \frac{(6.089)\log_{10}(6.089) - 0.47772393}{\log_{10}(6.089) + \log_{10} e}$$

$\Rightarrow$ $$x_2 = 6.089 - \frac{6.089 + 0.784460 - .47772303}{\log_{10} 6.089 + 0.434}$$

$$= 6.089 + \frac{1.0001387}{\log_{10} 6.089 + 0.434}$$

$$= 6.089 + (.0001387)(0.825) = 6.089114.$$

Example 14: ***Find the root of $x^2 - 5x + 2 = 0$ correct to the five decimal places by Newton-Raphson Method.*** **[B.C.A. (Agra) 2007, 2009; B.C.A. (Kurukshetra) 2010]**

Solution: The given equation has two real roots, and from the graph it can be verified that the approximate value of the roots are 0, 4.

Let $f(x) = x^2 - 5x + 2 \quad \Rightarrow \quad f'(x) = 2x - 5.$

By Newton's Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5x_n + 2}{2x_n - 5} = \frac{x_n{}^2 - 2}{2x_n - 5}$$

Putting $n = 0$, we get, $x_1 = \frac{-2}{-5} = .4$ [starting with $x_0 = 0$]

Again, we have $x_2 = \frac{(.4^2 - 2)}{2\,(.4) - 5} = \frac{-1.84}{-4.20} = .438095,$

$$x_3 = \frac{(.438095)^2 - 2}{2\,(.438095) - 5} = .438447 \text{ and } x_4 = \frac{(.438447)^2 - 2}{2\,(.438447) - 5} = .438447$$

From here it is clear that $x_3 = x_4$; that real root is .438447.

For the second real root, we start with $x_0 = 4.0$ and thus we have obtain

$$x_2 = \frac{(4)^2 - 2}{2\,(4) - 5} = \frac{14}{3}.$$

Further also,

$$x_2 = \frac{\left(\frac{14}{3}\right)^2 - 2}{2\left(\frac{14}{3}\right) - 5} = 4.564104\,; \quad x_3 = \frac{(4.564104)^2 - 2}{2\,(4.564104) - 5} = 4.561554,$$

$$x_4 = \frac{(4.561554)^2 - 2}{2\,(4.561554) - 5} = 4.561553; \text{ and } x_5 = \frac{(4.561553)^2 - 2}{2\,(4.561553) - 5} = 4.561553,$$

Obviously $x_5 = x_4$ so the real of $f(x) = 0$ is 4.561553.

Example 15: ***Find to four places of decimal the smaller root of the equation $e^{-x} - \sin x = 0$.***

[B.C.A. (Purvanchal) 2005, 2007]

Solution: We have $f(x) = e^{-x} - \sin x = 0$

Now $$x_{n+1} = x_n + \frac{e^{-x_n} - \sin x_n}{e^{-x_n} + \cos x_n} \qquad \text{...(1)}$$

$$\Rightarrow \quad x_1 = .6 + \frac{e^{\cdot 6} - \sin(.6)}{e^{\cdot 6} + \cos(.6)} = .6\;\frac{.54881 - .56464}{.54881 + .82524}$$

$$= .6 - \frac{.01583}{1.37415} = .6 - .01151 = .58848$$

Also $$x_2 = .58848 + \frac{e^{-.58848} - \sin(.58848)}{e^{-.58848} - \cos(.58848)}$$

$$= .58848 + \frac{.55519 - .55508}{.55519 + .8311} = .58848 + .000079287 = .588559287$$

Thus the desired root is $.5885$.

Example 16: ***By using Newton-Raphson Method find the root of $x^4 - x - 10 = 0$, which is nearer to $x = 2$, correct to three places of decimal.***

Solution: The approximate value of the root is given to be 2.

Now $$f(x) = x^4 - x - 10, \;\Rightarrow f'(x) = 4x^3 - 1$$

$$\therefore \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 10}{4x_n^3 - 1}$$

$$\Rightarrow \qquad x_1 = \text{first approximation}$$

$$= \frac{3x_0^4 + 10}{4x_0^3 - 1} = \frac{3.2^4 + 10}{4.2^3 - 1} = 1.871,$$

$$x_2 = \text{second approximation} = \frac{3x_1^4 + 10}{4x_1^3 - 1} = 1.85578 = 1.856.$$

$$x_3 = \text{third approximation},$$

$$= \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3\,(1.856)^4 + 10}{4\,(1.856)^3 - 1} = 1.856$$

Obviously $x_2 = x_3$ and hence the required root $= 1.856$.

Example 17: ***Using the staring value $x_0 = i$, find a zero of $x^4 + x^3 + 5x^2 + 4x + 4 = 0$.***

[B.C.A. (Agra) 2005]

Solution: By Newton's Method, we have

First approx, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = i - \frac{3\,i}{1 + 6\,i} = .486 + 0.919\,i$$

Second approximation, $$x_2 = 0.486 + 0.919\,i - \frac{f(0.486 + 0.919\,i)}{f'(0.486 + 0.919\,i)}$$

$$= 0.486 + 0.919\,i - \frac{-0.292 + 0.174\,i}{1.780 + 6.005\,i} = -0.499 + 0.866\,i$$

The actual root is $x = \dfrac{-1 + i\sqrt{3}}{2}$

Example 18: ***Find the cube root of*** **10.** **[B.C.A. (Rohilkhand) 2004, 2005, 2009]**

Solution: The Iterative formula for finding successive approximation of cube roots is given by

$$x_{n+1} = \frac{2x_n^3 + N}{3\,x_n^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right) \qquad ...(1)$$

$\because$ $(8)^{1/3} = 2$, $(27)^{1/3} = 3$, we may take initially $x_0 = 2.5$.

$\Rightarrow$ $$x_1 = \frac{1}{3}\left[2 \times 2.5 + \frac{10}{(2.5)^2}\right] = 2.2\,;\; x_2 = \frac{1}{3}\left[2 \times 2.2 - \frac{10}{(2.2)^2}\right] = 2.155,$$

and $$x_3 = \frac{1}{3}\left[2 \times 2.155 + \frac{10}{(2.155)^2}\right] = 2.15466.$$

Hence $\sqrt[3]{(10)} \approx 2.15466$.

Example 19: ***Apply (i) False Position Method (ii) Newton-Raphson Method to solve the equation*** $3x - \cos x - 1 = 0$. **[B.C.A. (Meerut) 2001, 2004, 2007]**

Solution: Let $f(x) = 3x - \cos x - 1 = 0$.

$$f(.60) = -.025 = \text{negative quantity},\; f(.61) = .010 = \text{positive quantity}.$$

$\therefore$ Real root lies in the interval $(.60, .61)$.

(i) False Position Method

$$\left\{\begin{array}{l} x_1 = .60,\ y_1 = -.025 \\ x_2 = .61,\ y_2 = .010 \end{array}\right. \Rightarrow x^{(1)} = x_1 + \frac{(x_2 - x_1)\,|\,y_1\,|}{|\,y_2\,| + |\,y_1\,|} = .60 + \frac{(.01)\,(.025)}{.025 + .010}$$

$$= .60 + \frac{(.01)\,(.025)}{.01 + .025} = .607$$

For the second approximation, we take

$$\left\{\begin{array}{l} x_1 = .607,\ y_1 = -.00036 \\ x_2 = .608,\ y_2 = \ .00321 \end{array}\right. \Rightarrow x^{(2)} = .607 + \frac{(.00321)\,(.00036)}{.00321 + .00036} = .6071$$

For the third approximation, we have

$$\left\{\begin{array}{l} x_1 = .6071,\ y_1 = -.00000 \\ x_2 = .6072,\ y_2 = \ .00035 \end{array}\right. \Rightarrow x^{(3)} = .6071 + \frac{(.0001)\,(.00000)}{.00000 + .00035} = .60710.$$

(ii) Newton's Raphson Method

For the first approximation, we take

$$f(x) = 3x - \cos x - 1 \qquad \Rightarrow \qquad f'(x) = 3 + \sin x$$

Now $$x_{n+1} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$\Rightarrow \qquad x_1 = .60 - \frac{3\,(.60) - \cos .60 - 1}{3 + \sin .60} \qquad [\text{starting from } x_0 = .60]$$

$$= .60 - \frac{1.8 - .82534 - 1}{3 + .56464} = .60710$$

Further, we have $$x_2 = .60710 - \frac{3\,(.60710) - \cos\,(.60710) - 1}{3 + \sin\,(.60710)}$$

$$= .60710 - \frac{1.82130 - 821321 - 1}{3 + .57070} = .60710$$

Obviously $x_1 = x_2$ so the real root of $f(x) = 0$ is $.60710$.

Example 20: ***Solve*** $\boldsymbol{x^4 - 5x^3 + 20x^2 - 40x + 60 = 0}$, ***by Newton-Raphson method given all the roots of the given equation are complex.***

Solution: Let $f(x) = x^4 - 5x^3 + 20x^2 - 40x + 60$, then

$$f'(x) = 4x^3 - 15x^2 + 40x - 40.$$

Now $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^4 - 5\,x_n^3 + 20\,x_n^2 - 40\,x_n + 60}{4\,x_n^3 - 15\,x_n^2 + 40\,x_n - 40} = \frac{3\,x_n^4 - 10\,x_n^3 + 20\,x_n^2 - 60}{4\,x_n^3 - 15\,x_n^2 + 40\,x_n - 40}$$

Starting with $x_0 = 2\,(1 + i)$, we get

$$x_1 = \frac{3\,(2 + 2\,i)^4 - 10\,(2 + 2\,i)^3 + 20\,(2 + 2\,i)^2 - 60}{4\,(2 + 2\,i)^3 - 15\,(2 + 2\,i)^2 + 40\,(2 + 2\,i) - 40} = \frac{-92\,i}{24\,i - 24}$$

$$= \frac{23}{12}\,(1 + i) = 1.92 + 1.92\,i, \qquad ...(1)$$

$$x_2 = \frac{3\,(1.92 + 1.92\,i)^4 - 10\,(1.92 + 1.92\,i)^3 + 20\,(1.92 + 1.92\,i) - 60}{4\,(1.92 + 1.92\,i)^3 - 15\,(1.92 + 1.92)^2 + 40\,(1.92 + 1.92) - 40}$$

$$= 1.915 + 1.908\,i \qquad ...(2)$$

Imaginary roots occur in pairs, so the roots of $f(x) = 0$ are $1.915 \pm 1.908\,i$ up to three places of decimal. Let the other roots of the given equation be $\alpha \pm i\beta$.

Then the sum of the roots is given by

$$(\alpha + i\beta) + (\alpha - i\beta) = (1.915 + 1.908\,i) + (1.915 - 1.908\,i) = -(-5) = 5 \qquad ...(3)$$

Also the product of roots is given by

$$(\alpha + i\beta)(\alpha - i\beta) = (\alpha^2 + \beta^2) = \{(1.95)^2 + (1.908)^2\} = 60 \qquad ...(4)$$

Solving (3) and (4), we get $\alpha = .585,\ \beta = 2.805$.

Hence, the other two roots are $(.585 \pm 2.805\, i)$.

Example 21: ***Solve the equation* log *x* = cos *x to five decimal places by Newton-Raphson Method.*** **[B.C.A. (Rohtak) 2004, 2007, 2013]**

Solution: Let $f(x) = \log_e x - \cos x = 0$.

Then it is clear that $f(x)$ is negative for $x = 1.3$ and is positive for $x = 1.4$, so the root lies between 1.3 and 1.4.

Now, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};\ n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{\log_e x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n} \qquad \left[\because\ f'(x) = \frac{1}{x} + \sin x\right]$$

$$\Rightarrow \quad x_1 = 1.3 - \frac{\log_e 1.3 - \cos(1.3)}{\frac{1}{1.3} + \sin(1.3)} \qquad [\text{starting from } x_0 = 1.3]$$

$$= 1.3 - \frac{(2.3026)(.11394) - .26750}{.76923 + .96356} = 1.3029.$$

Again putting $n = 2$, we get

$$x_2 = 1.3029 - \frac{\log_e 1.3029 - \cos(1.3029)}{\frac{1}{1.3029} + \sin(1.3029)} = 1.30295.$$

Finally, $$x_3 = 1.30295 - \frac{\log_e 1.30295 - \cos(1.30295)}{\frac{1}{1.30295} + \sin(1.30295)} = 1.30295$$

Obviously $x_2 = x_3$, Hence the root is 1.30295.

Example 22: ***Show that the modified Newton-Raphson's Method.***

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

Gives a quadratic convergence when the equation $f(x) = 0$ has a pair of double root in the neighbourhood of $x = x_n$.

[B.C.A. (Purvanchal) 2005, 2012; B.C.A. (Lucknow) 2007, 2013; B.C.A. (Kanpur) 2007, 2012]

Solution: $\varepsilon_{n+1} = \varepsilon_n - \dfrac{2f(a+\varepsilon_n)}{f'(a+\varepsilon_n)}$, where a, ε_n, ε_{n+1} have their usual meanings. Further expanding $f(a+\varepsilon_n)$ and $f'(a+\varepsilon_n)$ in powers of ε_n and using $f(a)=0$, $f'(a)=0$, since $x=a$ is a double root near $x = x_n$ we readily obtain

$$\varepsilon_{n+1} = \varepsilon_n - \frac{2\,\varepsilon_n^2\left[\dfrac{1}{2\,!}f''(a) + \dfrac{1}{3\,!}\varepsilon_n f'''(a) + \ldots\right]}{\varepsilon_n\left[f''(a) + \dfrac{\varepsilon_n}{2\,!}f'''(a) + \ldots\right]}$$

$$\simeq \varepsilon_n - \frac{2\,\varepsilon_n\left[\dfrac{1}{2\,!}f''(a) + \dfrac{1}{3\,!}\varepsilon_n f'''(a)\right]}{\varepsilon_n\left[f''(a) + \dfrac{\varepsilon_n}{2\,!}f'''(a)\right]}$$

$$\simeq \varepsilon_n^2 \frac{f'''(a)}{\left[f''(a) + \dfrac{\varepsilon_n f'''(a)}{2\,!}\right]} \simeq \frac{\varepsilon_n^2}{6}\,\frac{f'''(a)}{\left\{f''(a) + \dfrac{\varepsilon_n f'''(a)}{2}\right\}}$$

Thus $\varepsilon_{n+1} \simeq \dfrac{1}{6}\,\varepsilon_n^2\,\dfrac{f'''(a)}{f''(a)}$ implying that $\varepsilon_{n+1} \propto \varepsilon_n^2$ and hence the convergence is quadratic.

Problem Set

1. Evaluate (i) $\sqrt{(29)}$, (ii) $\left(\frac{1}{3}\right)^{1/4}$. **[B.C.A. (Avadh) 2012]**

2. Find the root of $\log x = \cos x$, by Newton's Method to five places of decimal.

3. Find $\sqrt{(30)}$ by using Iterative process. **[B.C.A. (Purvanchal) 2007]**

4. Determine the real root of $2x - 3\sin x - 5 = 0$, up to six decimal places, by Newton's Method.

5. Use Newton's Method for nearly equal roots to solve

$$x^3 - x^2 - 16.0001x - 19.9985 = 0$$

[B.C.A. (A.M.U.) 2010, 2013]

6. Solve $x^3 + 1.2x^2 - 4x - 4.8 = 0$ by False Position Method. **[B.C.A. (Agra) 2006]**

7. Find the p^{th} and reciprocal p^{th} root of a given number N by Newton-Raphson's Method. Hence obtain the cube root of 12.

8. Find the root of $x^2 - 5x + 2 = 0$ correct to five decimal places by Newton-Raphson Method. **[B.C.A. (Agra) 2003, 2008]**

9. Find a root of the equation $x^4 + x^3 - x^2 - 10x + 1 = 0$, which lies 0 and 1, correct to four places of decimal.

10. Find a root of a equation $x^3 + 29x - 97 = 0$, which lies between 2 and 3, correct to five places of decimal. **[B.C.A. (Meerut) 2005, 2008]**

Answers

1.	(i) 5.385165, (ii) 0.75
2.	1.3029
3.	5.47722
4.	2.883238
5.	$-2.990007,\ -2.310007$. Actual roots are $-1.99,\ -2.31, 5$
8.	4.561553
9.	0.3131
10.	2.68061

* * *

Unit

2

Chapter-2: Finite Differences

Chapter-3: Interpolation with Equal Intervals

Chapter-4: Interpolation with Unequal Intervals

Chapter-5: Central Difference Interpolation Formulae

Unit-2

Chapter 2

Finite Differences

2.1 Finite Differences

Let $y = f(x)$ be function where x can take the value $a, a+h, a+2h, \ldots, a+nh$ Let the corresponding value of y. *i.e.*, $f(x)$ be $f(a), f(a+h) \ldots. f(a+nh)$. The expression $f(a+h) - f(a)$ is called the **first difference** of $f(a)$ and we denote this difference by $\Delta f(a)$

Thus, $\Delta f(a) = f(a+h) - f(a)$

Similarly,

$f(a+2h) - f(a+h), \ldots., f(a+nh) - f\{a+(n-1)h\}$ are called the first differences and generally denoted as

$$\Delta f(x) = f(x+h) - f(x), x = a, a+h, \ldots., a+(n-1)h \qquad \ldots(1)$$

Here Δ is an operator and is called a **forward difference operator** or **difference operator**, h is called the **interval of differencing**.

If we operate Δ on (1), we get second differences

i.e., $\Delta\{\Delta f(x)\} = \Delta\{f(x+h) - f(x)\}$

or $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

or $\Delta^2 f(x) = \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}$ **from (1)**

$$= f(x+2h) - 2f(x+h) + f(x)$$

Here Δ^2 represents that the operation of differences has been done twice.

We may continue to define **third difference** as

$$\Delta^3 f(x) = \Delta\Delta^2 f(x) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$

$$= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x)$$

$$= \{f(x+3h) - f(x+2h)\} - 2\{f(x+2h) - f(x+h\} + \{f(x+h) - f(x)\}$$

$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

and so on.

In general the *n*th forward difference is given by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x).$$

From above we observe that each higher difference can be expressed in terms of the preceding difference.

2.1.1 Difference Table

The various differences obtained when shown in the form of a table is called a difference table. It is as given below:

Argument x	**Entry $f(x)$**	**First differences $\Delta f(x)$**	**Second differences $\Delta^2 f(x)$**
a	$f(a)$		
		$f(a+h) - f(a) = \Delta f(a)$	
$a+h$	$f(a+h)$		$\Delta f(a+h) - \Delta f(a) = \Delta^2 f(a)$
		$f(a+2h) - f(a+h)$ $= \Delta f(a+h)$	
$a+2h$	$f(a+2h)$		$\Delta f(a+2h) - \Delta f(a+h)$ $= \Delta^2 f(a+h)$
		$f(a+3h) - f(a+2h)$ $= \Delta f(a+2h)$	
$a+3h$	$f(a+3h)$		$\Delta f(a+3h) - \Delta f(a+2h)$ $= \Delta^2 f(a+2h)$
		$f(a+4h) - f(a+3h)$ $= \Delta f(a+3h)$	
$a+4h$	$f(a+4h)$		
....	....	....	....
....	....	....	....

The first entry $f(a)$ is called the leading term and the topmost difference in each column are called leading differences.

SOLVED EXAMPLES

Example 1: ***Given*** $f(0) = 3,\ f(1) = 12,\ f(2) = 81,\ f(3) = 200,\ f(4) = 100$ ***and*** $f(5) = 8$. ***Form a difference table and find*** $\Delta^5 f(0)$.

[B.B.A. (Meerut) 2010, 2012]

Solution: The differences table is as follow:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	3					
		9				
1	12		60			
		69		– 10		
2	81		50		– 259	
		119		– 269		755
3	200		– 219		496	
		– 100		227		
4	100		8			
		– 92				
5	8					

Hence, $\Delta^5 f(0) = 755$.

Example 2: ***If***

x :	1	2	3	4	5
y :	2	5	10	20	30

Find by forward difference table $\Delta^4 y(1)$.

Solution: The forward difference table as follows:

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	2				
		3			
2	5		2		
		5		3	
3	10		5		– 8
		10		– 5	
4	20		0		
		10			
5	30				

From table we observe that $\Delta^4 f(1) = -8$.

2.1.2 Backward Difference Operator ∇

The operator ∇ defined by

$\nabla f(x) = f(x) - f(x-h)$ is called **backward difference operator**.

The second backward difference of $f(x)$ denoted by $\nabla^2 f(x)$

i.e.,
$$\nabla^2 f(x) = \nabla\nabla f(x) = \nabla\{f(x) - f(x-h)\}$$
$$= \nabla f(x) - \nabla f(x-h)$$
$$= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\}$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

and so on.

In general the *n*th backward differences of $f(x)$ is given by

$$\nabla^n f(x) = \nabla^{n-1}[\nabla f(x)] = \nabla^{n-1}[f(x) - f(x-h)\}$$
$$= \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h).$$

2.1.3 Central Difference Operator δ

[B.C.A. (Agra) 2008]

The operator δ defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

is called the **central difference operator**.

2.1.4 Identity Operator I

The operator I defined by $If(x) = f(x)$, is called **identity operator**. The identity operator I is also often denoted by the symbol I.

2.1.5 The Shifting Operator E

[B.B.A. (Meerut) 2006; B.C.A. (Lucknow) 2010]

The operator E defined by $Ef(x) = f(x+h)$, where h is the interval of differencing is called the shifting operator or translation. By applying the operator E twice, we have

$$E^2 f(x) = EEf(x) = Ef(x+h) = f(x+2h),$$
$$E^3 f(x) = EE^2 f(x) = Ef(x+2h) = f(x+3h),$$

and so on.

In general $\quad E^n f(x) = f(x+nh).$

Also we define E^{-1} by

$$E^{-1} f(x) = f(x-h) \text{ so that } E^{-n} f(x) = f(x-nh).$$

2.1.6 Properties of the Operator E and Δ

1. The operators Δ and E are linear operators

   As $$\Delta[f_1(x)+f_2(x)] = \{f_1(x+h)+f_2(x+h)\}-\{f_1(x)+f_2(x)\}$$
   $$=\{f_1(x+h)-f_1(x)\}+\{f_2(x+h)-f_2(x)\}$$
   $$=\Delta f_1(x)+\Delta f_2(x)$$

   and $$\Delta[af(x)] = af(x+h)-af(x)$$
   $$=a[f(x+h)-f(x)\}=a\Delta f(x)$$

   Therefore, Δ is a linear operator.

   Similarly, we can show that E is a linear operator.

2. The operators Δ and E are commutative in operation with respect to constants

   If c is a constant, then
   $$\Delta\{cf(x)\}=cf(x+h)-cf(x)=c\,\Delta f(x)$$
   and $$E\{cf(x)\}=cf(x+h)=cEf(x).$$

3. The operators Δ and E are distributive

   We have,
   $$\Delta[f_1(x)+f_2(x)+\ldots.]=\Delta f_1(x)+\Delta f_2(x)+\ldots.$$
   and $$E[f_1(x)+f_2(x)+\ldots.]=Ef_1(x)+Ef_2(x)+\ldots.$$

4. The operators Δ and E are commutative
   $$(\Delta E)f(x)=\Delta[Ef(x)]=\Delta f(x+h)$$
   $$=f(x+2h)-f(x+h)=Ef(x+h)-Ef(x)$$
   $$=E[f(x+h)-f(x)]=(E\Delta)\,f(x)$$

   Thus, $$\Delta E = E\Delta.$$

5. The operators Δ and E are associative
   $$(\Delta E)\,\Delta f(x)=\Delta\,(E\Delta)\;f(x)$$

6. The operators Δ and E obey the law of indices
   $$\Delta^m\Delta^n f(x)=(\Delta\Delta\ldots.m\text{ times})\,(\Delta\Delta\ldots.n\text{ times})\;f(x)$$
   $$=[\Delta\Delta\ldots.(m+n)\text{ times}]\;f(x)$$
   $$=\Delta^{m+n}f(x)$$

   Similarly, $$E^mE^nf(x)=E^{m+n}f(x)$$

7. Difference of the product of two functions
   $$\Delta[f(x)\,g(x)]=[Ef(x)]\cdot\Delta g(x)+g(x)\cdot\Delta f(x)$$

   We have,
   $$\Delta[f(x)\cdot g(x)]=f(x+h)\cdot g(x+h)-f(x)\cdot g(x)$$

by definition of Δ

$$= f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$$

$$= f(x+h)\,[g(x+h) - g(x)] + g(x)\,[f(x+h) - f(x)]$$

$$= f(x+h)\cdot \Delta g(x) + g(x)\cdot \Delta f(x)$$

$$= Ef(x)\cdot \Delta g(x) + g(x)\cdot \Delta f(x)$$

8. Difference of the quotient of two functions

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot \Delta f(x) - f(x)\,\Delta g(x)}{g(x)\cdot Eg(x)}$$

We have,

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \left[\frac{f(x+h)}{g(x+h)}\right] - \left[\frac{f(x)}{g(x)}\right]$$

$$= \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{g(x)\cdot g(x+h)}$$

$$= \frac{[f(x+h) - f(x)]\,g(x) - [g(x+h) - g(x)]\,f(x)}{g(x)\cdot g(x+h)}$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot \Delta f(x) - f(x)\cdot \Delta g(x)}{g(x)\cdot Eg(x)}$$

9. $\Delta = E - I$ or $E = I + \Delta$

We have,

$$\Delta f(x) = f(x+h) - f(x)$$

$$= Ef(x) - If(x) = (E - I)\,f(x)$$

Therefore, $\Delta = E - I$

Also, $Ef(x) = f(x+h) = \{f(x+h) - f(x)\} + f(x)$

$$= \Delta f(x) + If(x)$$

$$= If(x) + \Delta f(x) = (I + \Delta)\,f(x)$$

So, that $E = I + \Delta$.

10. $\Delta^2 = E^2 - 2E + I$

We have,

$$\Delta^2 f(x) = \Delta\Delta f(x) = \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}$$

$$= E^2 f(x) - 2Ef(x) + If(x)$$

$$\Delta^2 f(x) = [E^2 - 2E + I]\, f(x)$$

$\therefore$ $$\Delta^2 = E^2 - 2E + I$$

Similarly, $$\Delta^3 = E^3 - 3E^2 + 3E - I.$$

11. $E^n = \sum_{r=0}^{n} \binom{n}{r} \Delta^r$ and $\Delta^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} E^r$

We have $E = I + \Delta$ so that $E^n = (I + \Delta)^n$

or $$E^n = \sum_{r=0}^{n} {}^nC_r\, I^{n-r} \Delta^r, \qquad \text{by binomial theorem}$$

or $$E^n = \sum_{r=0}^{n} \binom{n}{r} \Delta^r, \quad \because \quad I^{n-r} = I \quad \text{and} \quad I\,\Delta^r = \Delta^r$$

Again

$$\Delta = E - I \text{ so that } \Delta^n = (E - I)^n$$

or $$\Delta^n = [E + (-I)]^n = [(-I) + E]^n = \sum_{r=0}^{n} {}^nC_r\, (-I)^{n-r} E^r$$

by binomial theorem

or $$\Delta^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} I^{n-r} E^r = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} E^r$$

2.2 Fundamental Theorem of the Difference Calculus

[B.C.A. (Bhopal) 2007; B.C.A. (Agra) 2004, 2009; B.C.A. (Lucknow) 2011]

Theorem: If $f(x)$ be a polynomial of nth degree in x, then the nth difference of $f(x)$ is constant and $\Delta^{n+1} f(x) = 0$.

Proof: Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots. + a_n x^n$, where n is a positive integer and $a_0, a_1, \ldots., a_n$ are constant. Then by definition of Δ, we have

$$\begin{aligned}
\Delta f(x) &= f(x + h) - f(x) \\
&= [a_0 + a_1 (x + h) + a_2 (x + h)^2 + \ldots. + a_n (x + h)^n] \\
&\qquad - [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n] \\
&= a_1 h + a_2 [(x + h)^2 - x^2] + \ldots. + a_n [(x + h)^n - x^n] \\
&= a_1 h + a_2 [{}^2c_1 xh + h^2] + \ldots. \\
&\qquad + a_n [{}^nc_1 x^{n-1} h + {}^nc_2 x^{n-2} h^2 + \ldots. + {}^nc_n h^n] \\
&= b_0 + b_1 x + b_2 x^2 + \ldots. + b_{n-2} x^{n-2} + nha_n x^{n-1} \qquad \ldots(1)
\end{aligned}$$

Where $b_0, b_1, b_2, \ldots, b_{n-2}$ are constant coefficients. Thus, the first difference of a polynomial of degree n is again a polynomial of degree $n-1$.

Repeating the operation, we will observe that $\Delta\Delta\, f(x)$ *i.e.*, $\Delta^2 f(x)$ is a polynomial of degree $(n-2)$ with last term as $n(n-1)\, h^2 a_n x^{n-2}$.

Continuing this process n times, we will get a polynomial of degree $n-n$ *i.e.*, of degree zero with the last and the only term as

$$n(n-1)(n-2)\ldots 1\cdot h^n a_n x^{n-n} = n!\,h^n a_n.$$

Thus, $\Delta^n f(x) = n!\,h^n a_n$ which is a constant and hence $\Delta^{n+1} f(x) = 0$ and so all higher differences of order $> n$ are also zero.

Example 3: ***Evaluate the following:***

(i) $\Delta(x^2 + e^x + 2)$. **[B.C.A. (Rohilkhand) 2004]**

Solution: (i) By definition $\Delta f(x) = f(x+h) - f(x)$

$\therefore$ $$\Delta(x^2) = (x+h)^2 - x^2 = 2hx + h^2;$$

$$\Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1);$$

and $$\Delta(2) = 2\,\Delta(1) = 2\,(1-1) = 2\cdot 0 = 0$$

Hence, $$\Delta(x^2 + e^x + 2) = \Delta(x^2) + \Delta(e^x) + \Delta(2)$$
$$= 2hx + h^2 + e^x(e^h - 1) + 0$$

(ii) $$\Delta^2(ab^{cx}) = \Delta[\Delta ab^{cx}) = \Delta[ab^{c(x+h)} - ab^{cx}]$$
$$= \Delta[ab^{cx}(b^{ch} - 1)]$$
$$= \Delta[a(b^{ch} - 1)\, b^{cx}]$$
$$= a(b^{ch} - 1)\, b^{c(x+h)} - a(b^{ch} - 1)\, b^{cx}$$
$$= a(b^{ch} - 1)\, b^{cx}\,(b^{ch} - 1)$$
$$= a(b^{ch} - 1)^2 b^{cx}.$$

(iii) $$\Delta\left[\frac{2^x}{\underline{|(x+1)}}\right] = \frac{2^{x+1}}{\underline{|x+2}} - \frac{2^x}{\underline{|x+1}},$$ Taking the interval of differencing as unity

$$= \frac{2^x}{\underline{|x+1}}\left[\frac{2}{x+2} - 1\right]$$

$$= \frac{2^x}{\underline{|(x+1)}}\left[\frac{2-x-2}{x+2}\right] = \frac{-x\cdot 2^x}{\underline{|(x+2)}}$$

(iv) $\Delta \cot 2^x = \cot 2^{x+1} - \cot 2^x$, taking the interval of differencing as 1.

$$= \frac{\cos 2^{x+1}}{\sin 2^{x+1}} - \frac{\cos 2^x}{\sin 2^x}$$

$$= \frac{\cos 2^{x+1} \sin 2^x - \cos 2^x \sin 2^{x+1}}{\sin 2^{x+1} \sin 2^x} = \frac{\sin (2^x - 2^{x+1})}{\sin 2^x \cdot \sin 2^{x+1}}$$

$$= \frac{\sin \{2^x(1-2)\}}{\sin 2^x \sin 2^{x+1}} = \frac{-\sin 2^x}{\sin 2^x \cdot \sin 2^{x+1}} = -\operatorname{cosec} 2^{x+1}.$$

(v)
$$\frac{\Delta^2}{E} x^3 = \frac{(E-I)^2}{E} x^3 = \left[\frac{E^2 + I^2 - 2EI}{E}\right] x^3$$

$$= [E + I^2 E^{-1} - 2I]\, x^3$$

$$= Ex^3 + I^2 E^{-1} x^3 - 2Ix^3$$

$$= (x+1)^3 + (x-1)^3 - 2x^3$$

$$= x^3 + 1 + 3x^2 + 3x + x^3 - 1 - 3x^2 + 3x - 2x^3$$

$$= 6x$$

(vi)
$$\frac{\Delta^2 x^3}{Ex^3} = \frac{(E-I)^2 x^3}{Ex^3} = \frac{(E^2 + I - 2E)\, x^3}{Ex^3}$$

[B.C.A. (Rohilkhand) 2007]

$$= \frac{E^2 x^3 + Ix^3 - 2Ex^3}{Ex^3}$$

$$= \frac{(x+2)^3 + x^3 - 2\,(x+1)^3}{(x+1)^3}$$

$$= \frac{x^3 + 8 + 6x^2 + 12x + x^3 - 2\,(x+1)^3}{(x+1)^3}$$

$$= \frac{6\,(x+1)}{(x+1)^3}$$

$$= \frac{6}{(x+1)^2}$$

(vii) $\Delta \sinh (a + bx)$

$$= \sinh [a + (x+h)\, b] - \sinh [a + bx]$$

$$= 2 \cosh \frac{2a + 2bx + bh}{2} \sinh \frac{bh}{2}$$

$$= 2 \cosh \left(a + \frac{b}{2} + bx\right) \sinh \frac{b}{2}, \quad \text{for } h = 1$$

(viii) $\Delta \cosh (a + bx) = \cosh [a + b\,(x+h)] - \cosh [a + bx]$

$$= 2 \sinh \frac{2a + 2bx + bh}{2} \sinh \frac{bh}{2}$$

$$= 2 \sinh \left(a + \frac{b}{2} + bx\right) \sinh \frac{b}{2}, \text{ for } h = 1$$

Example 4: ***Prove that*** $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$***; the interval of differencing being h.***

Solution: We have, $Ee^x = e^{x+h}$

Again, $\Delta e^x = e^{x+h} - e^x$

$$= e^x (e^h - 1)$$

Similarly $\Delta^2 e^x = e^x (e^h - 1)^2$

$\therefore$ $$\left(\frac{\Delta^2}{E}\right) e^x = (\Delta^2 E^{-1})\, e^x$$

$$= \Delta^2 e^{x-h} = e^{-h} \Delta^2 e^x = e^{-h} e^x (e^h - 1)^2$$

$\therefore$ $$\text{R.H.S.} = \left(\frac{\Delta^2}{E}\right) e^x \left(\frac{Ee^x}{\Delta^2 e^x}\right) = e^{-h} e^x\, (e^h - 1)^2 \cdot \frac{e^{x+h}}{e^x\, (e^h - 1)^2} = e^x.$$

Example 5: ***Evaluate*** $\Delta^2 (3e^x)$.

Solution: We have, $\Delta\, (3e^x) = 3\Delta\, (e^x)$

$$= 3\, (e^{x+h} - e^x) = 3e^x (e^h - 1)$$

$\therefore$ $$\Delta^2 (e^x) = \Delta\, \{3e^x (e^h - 1)\} = 3\, (e^h - 1)\, (\Delta\, e^x)$$

$$= 3\, (e^h - 1)\, (e^{x+h} - e^x) = 3\, (e^h - 1)^2\, e^x$$

Example 6: ***Evaluate*** $\Delta\, (x + \cos x)$***; the interval of differencing is*** α.

[B.B.A. (Meerut) 2012]

Solution: We have, $\Delta\, (x + \cos x) = \Delta\, x + \Delta \cos x$

$$= \{(x + \alpha) - x\} + \{\cos\, (x + \alpha) - \cos\, x\}$$

$$= \alpha + 2 \sin \left(\frac{2x + \alpha}{2}\right) \sin \left(-\frac{\alpha}{2}\right)$$

$$= \alpha - 2 \sin \left(x + \frac{\alpha}{2}\right) \sin \frac{\alpha}{2}.$$

Example 7: ***Show that*** $\Delta \log\, f(x) = \log \left\{1 + \frac{\Delta\, f(x)}{f(x)}\right\}$.

Solution: We have, $\Delta \log\, f(x) = \log\, f(x + h) - \log\, f(x)$

$$= \log \left[\frac{f(x + h)}{f(x)}\right] = \log \left[\frac{Ef(x)}{f(x)}\right]$$

$$= \log \left[\frac{(1 + \Delta)\, f(x)}{f(x)}\right] = \log \left[\frac{f(x) + \Delta\, f(x)}{f(x)}\right]$$

$$= \log \left[1 + \frac{\Delta\, f(x)}{f(x)}\right].$$

Example 8: ***Evaluate*** $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right]$.

Solution: $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right] = \Delta^2\left[\dfrac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right]$

$$= \Delta^2\left[\frac{2}{x+2}+\frac{3}{x+3}\right]$$

$$= \Delta\cdot\Delta\left[\frac{2}{x+2}+\frac{3}{x+3}\right] \qquad \ldots(1)$$

Again, $\Delta\left[\dfrac{2}{x+2}+\dfrac{3}{x+3}\right] = \left[\dfrac{2}{x+3}-\dfrac{2}{x+2}\right]+\left[\dfrac{3}{x+4}-\dfrac{3}{x+3}\right]$

$$= -\frac{2}{(x+2)(x+3)}-\frac{3}{(x+4)(x+3)}$$

∴ from (1), we have,

$$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta\left[-\frac{2}{(x+2)(x+3)}-\frac{3}{(x+4)(x+3)}\right]$$

$$= -2\left[\frac{1}{(x+3)(x+4)}-\frac{1}{(x+2)(x+3)}\right]$$

$$-3\left[\frac{1}{(x+5)(x+4)}-\frac{1}{(x+4)(x+3)}\right]$$

$$= -2\left[\frac{(x+2)-(x+4)}{(x+2)(x+3)(x+4)}\right]-3\left[\frac{(x+3)-(x+5)}{(x+3)(x+4)(x+5)}\right]$$

$$= \frac{4}{(x+2)(x+3)(x+4)}+\frac{6}{(x+3)(x+4)(x+5)}.$$

Example 9: ***Find*** $\Delta^n\left(\dfrac{1}{x}\right)$.

Solution: We have, $\Delta^n\left(\dfrac{1}{x}\right) = \Delta^{n-1}\Delta\left(\dfrac{1}{x}\right)$

Now, $\Delta\left(\dfrac{1}{x}\right) = \dfrac{1}{x+1}-\dfrac{1}{x} = \dfrac{x-(x+1)}{x(x+1)} = \dfrac{(-1)}{x(x+1)}$

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\Delta\left(\frac{1}{x}\right) = \Delta\left\{\frac{(-1)}{x(x+1)}\right\} = (-1)\,\Delta\left\{\frac{1}{x(x+1)}\right\}$$

$$= (-1)\left\{\frac{1}{(x+1)(x+2)}-\frac{1}{x(x+1)}\right\}$$

$$= (-1)\frac{x-(x+2)}{x(x+1)(x+2)} = \frac{(-1)(-2)}{x(x+1)(x+2)}$$

Similarly, $$\Delta^3\left(\frac{1}{x}\right) = \frac{(-1)(-2)(-3)}{x(x+1)(x+2)(x+3)}$$

$$\Delta^n\left(\frac{1}{x}\right) = \frac{(-1)(-2)(-3)\ldots.(-n)}{x(x+1)(x+2)\ldots.(x+n)}$$

$$= \frac{(-1)^n \underline{|n}}{x(x+1)(x+2)\ldots.(x+n)}.$$

Example 10: ***Show that*** $\Delta^n \sin(a+bx) = \left(2\sin\frac{b}{2}\right)^n \sin\left[a+bx+n\left(\frac{b+\pi}{2}\right)\right].$

Solution: We have,

$$\Delta \sin(a+bx) = \sin[a+b(x+1)] - \sin(a+bx)$$

$$= 2\cos\left(\frac{a+bx+b+a+bx}{2}\right)\sin\left(\frac{a+bx+b-a-bx}{2}\right)$$

$$= 2\cos\left[a+bx+\frac{b}{2}\right]\sin\frac{b}{2}$$

$$= 2\sin\frac{b}{2}\sin\left[a+bx+\frac{b}{2}+\frac{\pi}{2}\right]$$

$$= 2\sin\frac{b}{2}\sin\left(a+bx+\frac{b+\pi}{2}\right)$$

and $$\Delta^2 \sin(a+bx) = \Delta\left[2\sin\frac{b}{2}\sin\left(a+bx+\frac{b+\pi}{2}\right)\right]$$

$$= 2\sin\frac{b}{2}\left[\sin\left\{a+b(x+1)+\frac{b+\pi}{2}\right\} - \sin\left(a+bx+\frac{b+\pi}{2}\right)\right]$$

$$= 2\sin\frac{b}{2}\cdot 2\cos\left(a+bx+\frac{2b+\pi}{2}\right)\sin\frac{b}{2}$$

$$= \left(2\sin\frac{b}{2}\right)^2 \cos\left(a+bx+\frac{2b+\pi}{2}\right)$$

$$= \left(2\sin\frac{b}{2}\right)^2 \sin\left[a+bx+2\left(\frac{b+\pi}{2}\right)\right]$$

Proceeding similarly, we have

$$\Delta^n \sin(a+bx) = \left(2\sin\frac{b}{2}\right)^n \sin\left[a+bx+n\left(\frac{b+\pi}{2}\right)\right]$$

Example 11: ***Show that*** $\Delta \cot(a+bx) = \dfrac{-\sin b}{\sin(a+bx)\sin(a+b+bx)}.$

Solution: We have,

$$\Delta \cot(a+bx) = \cot[a+b(x+1)] - \cot(a+bx)$$

$$= \frac{\cos[a+b(x+1)]}{\sin[a+b(x+1)]} - \frac{\cos(a+bx)}{\sin(a+bx)}$$

$$= \frac{\cos[a + bx + b]\sin(a + bx) - \cos(a + bx)\sin(a + bx + b)}{\sin(a + bx + b)\sin(a + bx)}$$

$$= \frac{-\sin b}{\sin(a + bx)\sin[a + bx + b]}.$$

Example 12: ***Prove that*** $\Delta \log x = \log\left(1 + \frac{h}{x}\right)$.

Solution: L.H.S. $= \Delta \log x$

$$= \log(x + h) - \log x$$

$$= \log \frac{x + h}{x}$$

$$= \log\left(1 + \frac{h}{x}\right)$$

Example 13: ***Prove that*** $\nabla = \Delta E^{-1} = E^{-1}\Delta = I - E^{-1}$.

Solution: We have,

$$\nabla f(x) = f(x) - f(x - h)$$

$$\Delta E^{-1} f(x) = \Delta f(x - h) = f(x) - f(x - h)$$

$$E^{-1}\Delta f(x) = E^{-1}[f(x + h) - f(x)]$$

$$= E^{-1} f(x + h) - E^{-1} f(x)$$

$$= f(x) - f(x - h);$$

$$(I - E^{-1})\, f(x) = f(x) - E^{-1} f(x)$$

$$= f(x) - f(x - h)$$

From above we observe

$$\nabla = \Delta E^{-1} = E^{-1}\Delta = I - E^{-1}.$$

Example 14: ***Find the successive difference of*** $f(x) = ab^{cx}$ ***and sum the first n differences.***

Solution: Here $f(x) = ab^{cx}$

$\therefore$ $\Delta f(x) = ab^{c(x+h)} - ab^{cx} = ab^{cx} \cdot (b^{ch} - 1)$

Again $\Delta^2 f(x) = \Delta \cdot \Delta f(x) = \Delta\,[ab^{cx} \cdot (b^{ch} - 1)]$

$$= a(b^{ch} - 1)\,\Delta b^{cx} = a(b^{ch} - 1)\,[b^{c(x+h)} - b^{cx}]$$

$$= a(b^{ch} - 1) \cdot b^{cx}(b^{ch} - 1) = ab^{cx}(b^{ch} - 1)^2$$

and so on

In general, $\Delta^n f(x) = ab^{cx}(b^{ch} - 1)^n$

Hence, the required sum

$$= \Delta f(x) + \Delta^2 f(x) + \dots + \Delta^n f(x)$$

$$= ab^{cx}(b^{ch}-1) + ab^{cx}(b^{ch}-1)^2 + \dots + ab^{cx}(b^{ch}-1)^n$$

$$= ab^{cx}(b^{ch}-1)\,[1 + (b^{ch}-1) + (b^{ch}-1)^2 + \dots + (b^{ch}-1)^{n-1}]$$

$$= ab^{cx}(b^{ch}-1)\left[\frac{1-(b^{ch}-1)^n}{1-(b^{ch}-1)}\right] = ab^{ch}(b^{ch}-1)\,\frac{(b^{ch}-1)^n-1}{b^{ch}-2}.$$

Example 15: Evaluate $\boldsymbol{\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2\sin(x+h)}{E\sin(x+h)}}$.

Solution: We have, $\frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^2\sin(x+h)}{E\sin(x+h)}$

$$= \frac{(E-1)^2}{E}\sin(x+h) + \frac{(E-1)^2\sin(x+h)}{E\sin(x+h)}$$

$$= (E-2+E^{-1})\sin(x+h) + \frac{(E^2-2E+1)\sin(x+h)}{E\sin(x+h)}$$

$$= [\sin(x+2h) - 2\sin(x+h) + \sin x] + \frac{[\sin(x+3h) - 2\sin(x+2h) + \sin(x+h)]}{\sin(x+2h)}$$

$$= 2\sin(x+h)(\cos h - 1) + \frac{2\sin(x+2h)(\cos h-1)}{\sin(x+2h)}$$

$$= 2(\cos h - 1)[\sin(x+h) + 1].$$

2.2.1 Relation between Differential Operator D and Difference Operator Δ

Let $f(x)$ be any function. Then by definition, we have

$$Df(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \dots(1)$$

Also by Taylor's theorem, we have

$$f(x+h) = f(x) + hf'(x) = \frac{h^2}{\lfloor 2}f''(x) + \dots$$

$$= f(x) + h\cdot Df(x) + \frac{h^2}{\lfloor 2}D^2 f(x) + \dots$$

$$= \left(1 + hD + \frac{h^2D^2}{\lfloor 2} + \dots\right)f(x)$$

$$= e^{hD}f(x) \qquad \dots(2)$$

Also by calculus of finite differences, we have

$$f(x+h) = Ef(x) \qquad \dots(3)$$

From (2) and (3), we get $e^{hD} \equiv E \equiv 1 + \Delta$

or $$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 \;.....$$

or $$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} \;....\right]$$

2.3 Factorial Notation

The continued product of n factors

$$x\,(x-h)\,(x-2h)\,....\,\{x-(n-1)\,h\}$$

in which the first factor is x and the successive factors decrease by a constant difference (say h) is called a factorial function and is denoted by $x^{(n)}$, where n is a positive integer.

Thus,

$$x^{(1)} = x;\; x^{(2)} = x\,(x-h);\; x^{(3)} = x\,(x-h)\,(x-2h) \text{ and so on in general}$$

$$x^{(n)} = x\,(x-h)\,(x-2h)\,....\,\{x-(n-1)\,n\}$$

As a particular call when $h = 1$, we define

$$x^{(1)} = x;\; x^{(2)} = x\,(x-1);\; x^{(3)} = x\,(x-1)\,(x-2)\,....$$

And $$x^{(n)} = x\,(x-1)\,(x-2)\,....\,\{x-(n-1)\}$$

$$= \frac{x\,(x-1)\,(x-2)\,....\,\{x-(n-1)\}\cdot(x-n)!}{(x-n)!}$$

$$= \frac{\lfloor\underline{x}}{\lfloor\underline{x-n}},\; x > n.$$

The function $f(x) = x^{(n)} = x\,(x-h)\,....\,\{x-(n-1)\,h\}$, $n > 0$ is called the **factorial polynomial of degree n.**

2.3.1 Differences of a Factorial Polynomial

Theorem: To prove that $\Delta^n x^{(n)} = \lfloor\underline{n}\;h^n$

and $$\Delta^{n+1} x^{(n)} = 0.$$

Proof: As defined we have,

$$x^{(n)} = x\,(x-h)\,(x-2h)\,....\,\{x-(n-1)\,h\}$$

So that, $$(x+h)^{(n)} = (x+h)\cdot x\,(x-h)\,(x-2h)\,....\,\{x-(n-2)\,h\}.$$

Now, $$\Delta\, x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)\,x\,(x-h)\,(x-2h)\,....\,\{(x)-(n-2)h\} - [x\,(x-h)\,(x-2h)\,...\,x-(n-1)\,h]$$

$$= x\,(x-h)\,....\,\{x-(n-2)\,h\}\,[x+h-\{x-(n-1)\,h\}]$$

$$= x^{(n-1)}\cdot nh = nh\,x^{(n-1)} \qquad ...(1)$$

when is a polynomial of degree $(n-1)$ in factorial notation.

Again, $$\Delta^2 x^{(n)} = \Delta\Delta\, x^{(n)} = \Delta\,[nhx^{(n-1)}], \qquad \text{from (1)}$$

$$= nh\,\Delta\, x^{(n-1)} = nh\,[(x+h)^{n-1)} - x^{(n-1)}]$$

$$= nh\,[(x+h)\cdot x\,(x-h)\,\ldots.\{x-(n-3)h\} - x\,(x-h)\,(x-2h)\,\ldots. \{x-(n-2)\,h\}]$$

$$= nh\cdot x\,(x-h)\,\ldots.\{x-(n-3)\,h\}\,[x+h-\{x-(n-2)\,h\}]$$

$$= nh\cdot x^{(n-2)}\cdot(n-1)h = n\,(n-1)h^2\,x^{(n-2)} \qquad \ldots(2)$$

which is a polynomial of degree $(n-2)$ in factorial notation proceeding in this way, we get

$$\Delta^n x^{(n)} = n\,(n-1)\,\ldots.\,1\cdot h^n\cdot x^{n-n} = \lfloor\underline{n}\,h^n x^0\ \ldots.$$

$$= \lfloor\underline{n}\,h^n \qquad \ldots(3)$$

which is a constant.

Hence, $$\Delta^{n+1} x^{(n)} = \Delta\,(\Delta^n x^n) = \Delta\,[\lfloor\underline{n}\,h^n] = \Delta\,(\text{constant}) = 0$$

$$\Delta^k x^{(n)} = 0 \quad \text{for } k > n$$

In particular when $h = 1$, $\Delta^n x^{(n)} = \lfloor\underline{n}$.

2.3.2 To Show that $x^{(-n)} = \dfrac{1}{(x+n)^{(n)}}$ the Interval of Differencing being Unity

[B.C.A. (Kanpur) 2009]

Proof: By definition of $x^{(n)}$ for $h = 1$, we have

$$x^{(n)} = x\,(x-1)\,(x-2)\,\ldots.\{x-(n-2)\}\,\{x-(n-1)\}$$

or $$x^{(n)} = x^{(n-1)}\,\{x-(n-1)\}$$

For $$n = 0,\ x^{(0)} = x^{(-1)}\,(x+1)$$

or $$x^{(-1)} = \frac{1}{x+1}, \qquad \because \quad x^{(0)} = 1 \qquad \ldots(1)$$

and for $$n = -1,\ x^{(-1)} = x^{(-2)}(x+2)$$

or $$x^{(-2)} = \frac{x^{(-1)}}{x+2} = \frac{1}{(x+1)\,(x+2)} = \frac{1}{(x+2)^{(2)}}$$

Similarly, $$x^{(-3)} = \frac{1}{(x+1)\,(x+2)\,(x+3)} = \frac{1}{(x+3)^{(3)}}$$

Thus, $$x^{(-n)} = \frac{1}{(x+1)\,\ldots.\,(x+n)} = \frac{1}{(x+n)^{(n)}} \qquad \ldots(2)$$

In general if the interval of differencing is h, then

$$x^{(-n)} = \frac{1}{(x+h)\,\ldots.\,(x+nh)} = \frac{1}{(x+nh)^{(n)}} \qquad \ldots(3)$$

Now, $$\Delta x^{(-n)} = \Delta \frac{1}{(x+nh)^{(n)}} = \frac{1}{(x+h+nh)^{(n)}} - \frac{1}{(x+nh)^{(n)}}$$

$$= \frac{1}{\{x+(n+1)h\}\{x+nh\}\ldots.(x+2h)} - \frac{1}{(x+nh)\ldots.(x+2h)(x+h)}$$

$$= \frac{(x+h)-\{x+(n+1)h\}}{\{x+(n+1)h\}\{x+nh\}\ldots.\{x+h\}}$$

$$= \frac{-nh}{(x+h)\ldots.(x+nh)\{x+(n+1)h\}} = -nh \cdot x^{(-[n+1])}, \qquad \text{from (3)}$$

or $$\Delta x^{(-n)} = (-n)\cdot hx^{(-[n+1])} \qquad \ldots(4)$$

or $$\frac{\Delta}{\Delta x} x^{(-n)} = (-n)\, x^{(-[n+1])}, \qquad \because \; h = \Delta x$$

Similarly, $$\Delta^2 x^{(-n)} = \Delta[\Delta x^{(-n)}]$$

$$= \Delta[-nhx^{(-[n+1])}] = -nh\,\Delta\, x^{(-[n+1])}$$

$$= (-nh)[-(n+1)h]\,x^{(-[n+2])}$$

$$\Delta^2 x^{(-n)} = (-1)^2 n(n+1)h^2 x^{(-[n+2])}$$

and so on.

To find $\Delta^{-1}x^{(n)}$.

We have, $$\Delta x^{(2)} = 2\cdot hx^{(1)}, \quad \because \; \Delta x^{(n)} = nhx^{(n-1)}$$

or $$\Delta \frac{x^{(2)}}{2h} = x^{(1)}$$

or $$\Delta^{-1} x^{(1)} = \frac{x^{(2)}}{2h} + t(x)$$

where $t(x)$ is a periodic function of period h

Similarly $$\Delta^{-1} x^{(2)} = \frac{x^{(3)}}{3h} + t(x).$$

In general $$\Delta^{-1} x^{(n)} = \frac{x^{(n+1)}}{(n+1)h} + t(x).$$

Example 16: ***Find the value of***

(*i*) $\frac{\Delta}{\Delta x}[2x^{(5)}]$ **(*ii*)** $\Delta[9x^{(4)}]$

(*iii*) $\frac{\Delta}{\Delta x}[3x^{(2)}]$ **(*iv*)** $\Delta[-5x^{(-4)}]$

Solution: Let the interval of differencing be h. then,

(i) $$\frac{\Delta}{\Delta x}[2x^{(5)}] = 5\cdot 2x^{(4)} = 10x^{(4)}$$

$$= 10x(x-h)(x-2h)(x-3h)$$

(ii) $$\Delta\,[9x^{(4)}] = 4\cdot 9\cdot hx^{(3)} = 36hx^{(3)}$$

$$= 36hx\,(x-h)\,(x-2h)$$

(iii) $$\frac{\Delta}{\Delta x}\,[3x^{(-2)}] = (-2)\cdot 3x^{(-3)}$$

$$= -6x^{(-3)} = \frac{-6}{(x+h)\,(x+2h)\,(x+3h)}$$

(iv) $$\Delta\,[-5(x)^{(-4)}] = (-4)\cdot(-5)\;hx^{(-5)} = 20hx^{(-5)}$$

$$= \frac{20h}{(x+h)\,(x+2h)\,(x+3h)\,(x+4h)\,(x+5h)}$$

Example 17:* *Prove that

(i) $\boldsymbol{\Delta x^{(n)} = nx^{(n-1)}}$ ***(ii)*** $\boldsymbol{\Delta^2 x^{(n)} = n\,(n-1)\,x^{(n-2)}}$

Solution: (i) We have, $x^{(n)} = x\,(x-1)\,(x-2)\,\ldots.\,[x-(n-1)]$

$$\Delta x^{(n)} = (x+1)\,x\,(x-1)\,(x-2)\,\ldots.\,[x-(n-2)] - x\,(x-1)\,(x-2)\,\ldots.\,[x-(n-1)]$$

$$= x\,(x-1)\,(x-2)\,\ldots.\,[x-(n-2)]\,\{x+1-x+n-1\}$$

$$= x^{(n-1)}n = nx^{(n-1)}$$

(ii) $$\Delta^2 x^{(n)} = \Delta\Delta\,x^{(n)} = \Delta nx^{(n-1)}$$

$$= n\,(n-1)\,x^{(n-2)}.$$

Example 18:* *Express the function $\boldsymbol{2x^3 - 3x^2 + 3x - 10}$ ***in factorial notation, the interval of differencing being unity. Also find its successive difference.***

Solution: Let $2x^3 - 3x^2 + 3x - 10 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D.$

The coefficient A, B, C, D are easily obtained by the method of synthetic division.

The procedure is given below:

1	2	–3	3	$-10 = D$
	0	2	–1	
2	2	–1	$2 = C$	
	0	4		
3	2	$3 = B$		
	0			
	$2 = A$			

Hence, $$f(x) = 2x^{(3)} + 3x^2 + 2x^{(1)} - 10$$

$\therefore \qquad \Delta f(x) = 2\cdot 3x^{(2)} + 3\cdot 2x^{(1)} + 2$

$$= 6x^{(2)} + 6x^{(1)} + 2$$

$$= 6x\,(x-1) + 6x + 2 = 6x^2 + 2$$

$$\Delta^2 f(x) = 6\cdot 2x^{(1)} + 6 = 12x + 6$$

and $\qquad \Delta^3 f(x) = 12$.

Example 19: ***A second degree polynomial passes through* (0, 1), (1, 3), (2, 7) *and* (3, 13). *Find the polynomial.*** **[B.C.A. (Lucknow) 2011]**

Solution: Let us consider the 2nd degree polynomial in factorial notation

i.e., Let $\qquad f(x) = Ax^{(2)} + Bx^{(1)} + C.$

Now, the point $(0,1) \Rightarrow x = 0,\ y = 1 \quad i.e.,\ f(x) = 1;$

$$(1,3) \Rightarrow x = 1,\ f(x) = 3;\ (2,7) \Rightarrow x = 2,\ f(x) = 7$$

and $\qquad (3,13) \Rightarrow x = 3,\ f(x) = 13$

Thus, the difference table for the given values of x and $f(x)$ is as follow:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

Now $\qquad f(x) = Ax^{(2)} + Bx^{(1)} + C$

$\therefore \qquad \Delta f(x) = 2Ax^{(1)} + B = 2Ax + B$

Hence, $\qquad \Delta f(0) = B$

But, $\qquad \Delta f(0) = 2$ so that $B = 2$

Also $\qquad \Delta^2 f(x) = 2A \quad \therefore\ \Delta^2 f(0) = 2A.$ But $\Delta^2 f(0) = 2$

$\therefore \qquad 2A = 2 \Rightarrow A = 1$

Again for $\qquad x = 0,\ f(0) = C$ giving $C = 1,$

Thus, $\qquad f(x) = 1\cdot x^{(2)} + 2\cdot x^{(1)} + 1$

$$= x\,(x-1) + 2x + 1$$

$$= x^2 + x + 1.$$

Example 20: ***A third degree polynomial passes through the points* (0, – 1); (1, 1), (2, 1) *and* (3, – 2). *Find the polynomial.***

Solution: Let $\quad f(x) = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$

The difference table for the given values of x and $f(x)$ is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	– 1			
		2		
1	1		– 2	
		0		–1
2	1		– 3	
		– 3		
3	– 2			

Now $\quad \Delta f(x) = 3Ax^{(2)} + 2Bx^{(1)} + C$

$$= 3Ax(x-1) + 2Bx + C$$

$\therefore \quad \Delta f(0) = C$ and from table $\Delta f(0) = 2$

Comparing the two values, we get $C = 2$

Now, $\quad \Delta^2 f(x) = 6Ax^{(1)} + 2B = 6Ax + 2B$

$\therefore \quad \Delta^2 f(0) = 2B$ and $\Delta^2 f(0) = -2$ (from table)

$\therefore \quad 2B = -2$ giving $B = -1$

Again $\quad \Delta^3 f(x) = 6A \quad \therefore \quad \Delta^3 f(0) = 6A$

and from table $\Delta^3 f(0) = -1$

Equating the two values, we get

$$6A = -1 \Rightarrow A = -1/6$$

Also $f(0) = D$ and from table $f(0) = -1$.

Comparing the value of A, B, C and B, the required polynomial

$$f(x) = -\frac{1}{6}x^{(3)} - 1\cdot x^{(2)} + 2x^{(1)} - 1$$

$$= -\frac{1}{6}x(x-1)(x-2) - x(x-1) + 2x - 1$$

$$f(x) = -\frac{1}{6}[x^3 + 3x^2 - 16x + 6]$$

Example 21: ***Calculate $\Delta^{-1}x^3$.***

Solution: Let $\quad x^3 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$

or $\quad x^3 = Ax(x-h)(x-2h) + Bx(x-h) + Cx + D$

or $\quad x^3 = Ax^3 + (B - 3Ah)x^2 + (2Ah^2 - Bh + C)x + D$

Equating coefficients of like powers of x,

$$A = 1, B = 3hA, 2Ah^2 - Bh + C = 0, D = 0$$

i.e., $$A = 1, B = 3h, C = h^2, D = 0$$

$\therefore$ $$x^3 = x^{(3)} + 3hx^{(2)} + h^2 x^{(1)}$$

$\therefore$ $$\Delta^{-1}x^3 = \Delta^{-1}x^{(3)} + 3h\Delta^{-1}x^{(2)} + h^2\Delta^{-1}x^{(1)}$$

$$= \left\{\frac{x^{(4)}}{4h} + t_1(x)\right\} + 3h\left\{\frac{x^{(3)}}{3h} + t_2(x)\right\} + h^2\left(\frac{x^{(2)}}{2h} + t_3(x)\right\}$$

$$\Delta^{-1}x^3 = \frac{x^{(4)}}{4h} + x^{(3)} + \frac{h}{2}x^{(2)} + t(x)$$

where, $t(x)$ is a period function of period h.

Example 22: ***Obtain the function whose first difference is***

$$\boldsymbol{x^3 + 4x^2 + 9x + 12.}$$

Solution: Converting the given function into factorial notation.

We have, $$x^3 + 4x^2 + 9x + 12 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

By synthetic division method, we get

$$\begin{array}{c|cccl}
1 & 1 & 4 & 9 & 12 = D \\
 & 0 & 1 & 5 & \\ \cline{2-4}
2 & 1 & 5 & 14 = C & \\
 & 0 & 2 & & \\ \cline{2-3}
3 & 1 & 7 = B & & \\
 & 0 & & & \\ \cline{2-2}
 & 1 = A & & &
\end{array}$$

Thus, the given function $= x^{(3)} + 7x^{(2)} + 14x^{(1)} + 12$.

As given $$\Delta f(x) = x^{(3)} + 7x^{(2)} + 14x^{(1)} + 12$$

$\therefore$ $$f(x) = \Delta^{-1}x^{(3)} + 7\Delta^{-1}x^{(2)} + 14\Delta^{-1}x^{(1)} + \Delta^{-1}(12)$$

$$= \frac{x^{(4)}}{4} + 7\cdot\frac{x^{(3)}}{3} + 14\cdot\frac{x^{(2)}}{2} + 12\cdot\frac{x^{(1)}}{1} + k$$

where k is a constant and the interval of differencing has been taken as 1.

or $$f(x) = \frac{x(x-1)(x-2)(x-3)}{4} + \frac{7x(x-1)(x-2)}{3} + 7x(x-1) + 12x + k$$

$$f(x) = \frac{x^4}{4} + \frac{5x^3}{6} - \frac{7x^2}{4} + \frac{49x}{6} + k$$

2.3.3 Generalized Factorial Functions

Let $f(x)$ be any function, then generalized factorial function is defined by

$$[f(x)]^{(n)} = f(x)\ f(x-h)\ f(x-2h) \ldots.\ f[x-(n-1)h] \quad \text{for } n = 1, 2, 3, \ldots$$

and $$[f(x)]^{(-n)} = \frac{1}{f(x+h)\ f(x+2h) \ldots.(f(x+nh)}, n = 1, 2, 3, \ldots.$$

Example 23: ***Find*** $(ax+b)^{(n)}$ ***and*** $(ax+b)^{(-n)}$.

Solution: Here, $f(x) = ax + b$

$\therefore$ $$f(x-h) = a\,(x-h) + b = ax + b - ah$$

$$f(x-2h) = a\,(x-2h) + b = ax + b - 2ah;$$

. . . .

$$f\{x-(n-1)\,h\} = a\,\{x-(n-1)\,h\} + b = ax + b - nah + ah$$

Hence, by definition,

$$(ax+b)^{(n)} = (ax+b)\,(ax+b-ah)\,(ax+b-2ah) \ldots.(ax+b-nah+ah)$$

and $$(ax+b)^{(-n)} = \frac{1}{(ax+b+ah) \ldots.(ax+b+nah)}$$

2.4 Differences of Zero

For the operators Δ and E, we have

$$\Delta = E - I \qquad \therefore \quad \Delta^n = (E-I)^n$$

If $$f(x) = x^m, \text{ we have}$$

$$\Delta^n x^m = (E-I)^n x^m$$

$$= [E^n - {}^nC_1 E^{n-1} I + {}^nC_2 E^{n-2} I^2 \ldots. + (-1)^n I^n]\ x^m$$

$$= E^n x^m - {}^nC_1 E^{n-1} x^m + {}^nC_2 E^{n-2} x^m - \ldots. + (-1)^n I^n x^m$$

$$= [x+n]^m - {}^nC_1 [x+n-1]^m + {}^nC_2\ [x+n-2]^m - \ldots.. + (-1)^n x^m$$

Taking interval of differencing $h = 1$.

This expression for $x = 0$, becomes

$$[\Delta^n x^m]_{x=0} = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots.$$

$$+ {}^nC_{n-1} (-1)^{n-1} + 0 \qquad \ldots(1)$$

The expression (1) for $[\Delta^n x^m]_{x=0}$ is written as $\Delta^n 0^m$ and is known as differences of zero.

$\therefore$ $$\Delta^n 0^m = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots. + {}^nC_{n-1} (-1)^{n-1}$$

If $n = 1, m = 1$ then $\Delta^1 0^1 = 1^1 = 1,$

$n = 2, m = 2$ then $\Delta^2 0^2 = 2^2 - 2 \cdot (2-1)^2 = 2 = \underline{|2}$

and $n = 3, m = 3 \Rightarrow \Delta^3 0^3 = 3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 = 6 = \underline{|3}$

Proceeding in this way, we observe that $\Delta^n 0^n = \underline{|n}$

Also $\Delta^{n+1} 0^n = 0$ *i.e.*, $\Delta^n 0^m = 0$ for $n > m$.

Example 24: ***Calculate the value of***

(i) $\boldsymbol{\Delta^2 0^5}$ ***(ii)*** $\boldsymbol{\Delta^3 0^6}$ ***(iii)*** $\boldsymbol{\Delta^4 0^5}$

(iv) $\boldsymbol{\Delta^5 0^5}$ ***(v)*** $\boldsymbol{\Delta 0^n}$.

Solution: (i) $\Delta^2 0^5 = 2^5 - {}^2C_1 (2-1)^5 + {}^2C_2 (2-2)^5$

$= 32 - 2 \times 1 = 30;$

(ii) $\Delta^3 0^6 = 3^6 - {}^3C_1 (3-1)^6 + {}^3C_2 (3-2)^6 - 0$

$= 729 - 3 \cdot 64 + 3 \cdot 1 = 540$

(iii) $\Delta^4 0^5 = 4^5 - {}^4C_1 (4-1)^5 + {}^4C_2 (4-2)^5 - {}^4C_3 \ (4-3)^5 + 0$

$= 1024 - 4 \cdot 243 + 6 \cdot 32 - 4 \cdot 1$

$= 240$

(iv) $\Delta^5 0^5 = 5^5 - {}^5C_1 4^5 + {}^5C_2 3^5 - {}^5C_3 2^5 + {}^5C_4 1^5$

$= 3125 - 5 \times 1024 + 10 \times 243 - 10 \times 32 + 5$

$= 3125 - 5120 + 2430 - 320 + 5$

$= 5560 - 5440 = 120$

(v) $\Delta 0^n = 1^n - {}^1C_1 (1-1) = 1$

2.4.1 To Prove the Recurrence Relation

$$\Delta^n 0^m = n\,[\Delta^{n-1} 0^{m-1} + \Delta^n 0^{m-1}]$$

Proof: We have, $\Delta^n 0^m = n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots. + {}^nC_{n-1} (-1)^{n-1}$

$$= n^m - n\,(n-1)^m + \frac{n\,(n-1)}{\underline{|2}} (n-2)^m - \ldots + n\,(-1)^{n-1}$$

$$= n\,[n^{m-1} - (n-1)^m + \frac{(n-1)\,(n-2)^m}{\underline{|2}} - \ldots + (-1)^{n-1}]$$

$$= n\,[n^{m-1} - (n-1)\,(n-1)^{m-1} + \frac{(n-1)\,(n-2)}{\underline{|2}} (n-2)^{m-1} - \ldots.$$

$$+ (-1)^{n-1}]$$

$$= n\,[\{1 + (n-1)\}^{m-1} - {}^{(n-1)}C_1\,(n-1)^{m-1} + {}^{(n-1)}C_2\,(n-2)^{m-1} - \ldots. + (-1)^{n-1}]$$

$$= n\,[\{1 + (n-1)\}^{m-1} - {}^{(n-1)}C_1\,\{1 + (n-2)\}^{m-1} + {}^{(n-1)}C_2\{1 + (n-3)\}^{m-1} - \ldots. + (-1)^{n-1}]$$

$$= n\,[E^{n-1}(1)^{m-1} - {}^{(n-1)}C_1 E^{n-2}(1)^{m-1} + {}^{(n-1)}C_2\ \ E^{n-3}(1)^{m-1} - \ldots. + (-1)^{n-1}(1)^{m-1}]$$

$$= n\,[E^{n-1} - {}^{(n-1}C_1 E^{n-2} + {}^{(n-1)}C_2 E^{n-3} - \ldots. + (-1)^{n-1}]\,(1)^{m-1}$$

$$= n\,[E - I]^{n-1}(1)^{m-1} = n\Delta^{n-1}(1)^{m-1}$$

$$= n\Delta^{n-1}E0^{m-1} \qquad [\because \text{For } h = 1,\ Ex^{m-1} = (x+1)^{m-1} \Rightarrow E0^{m-1} = (1)^{m-1}]$$

$$= n\Delta^{n-1}(I + \Delta)\,0^{m-1}$$

$$= n\,[\Delta^{n-1}I + \Delta^{n}]\,0^{m-1}$$

$$= n\,[\Delta^{n-1}0^{m-1} + \Delta^{n}0^{m-1}]$$

which is the required recurrence relation between difference of zero for different values of n and m.

Example 25: ***Prove that*** $\boldsymbol{\Delta^{n}0^{n+1} = \dfrac{n\,(n+1)}{2}\,\Delta^{n}0^{n}}$.

Solution: We have, $\Delta^{n}0^{m} = n\,[\Delta^{n-1}0^{m-1} + \Delta^{n}0^{m-1}]$

for $m = n + 1$, this relation becomes.

$$\Delta^{n}0^{n+1} = n\,[\Delta^{n-1}0^{n} + \Delta^{n}0^{n}]$$

$$= n\,[\Delta^{n}0^{n} + \Delta^{n-1}0^{n}] \qquad \ldots(1)$$

In this relation replacing n by $(n-1)$, we get

$$\Delta^{n-1}0^{n} = (n-1)\,[\Delta^{n-1}0^{n-1} + \Delta^{n-2}0^{n-1}]$$

Repeating the process of replacement, we get

$$\Delta^{n-2}0^{n-1} = (n-2)\,[\Delta^{n-2}0^{n-2} + \Delta^{n-3}0^{n-2}]$$

. . . .

$$\Delta^{2}0^{3} = 2\,[\Delta^{2}0^{2} + \Delta^{1}0^{2}]$$

$$\Delta 0^{2} = 1\,[\Delta^{1}0^{1} + \Delta^{0}0^{1}]$$

From (1) by back substitutions of there values, we have,

$$\Delta^{n}0^{n+1} = n\,[\Delta^{n}0^{n} + (n-1)\,\{\Delta^{n-1}0^{n-1} + \Delta^{n-2}0^{n-1}\}]$$

$$= n\Delta^{n}0^{n} + n\,(n-1)\,\Delta^{n-1}0^{n-1} + n\,(n-1)\,\{(n-2)[\Delta^{n-2}0^{n-2} + \Delta^{n-3}0^{n-2}]\}$$

$$= n\Delta^{n}0^{n} + n\,(n-1)\,\Delta^{n-1}0^{n-1} + n\,(n-1)\,(n-2)\,\Delta^{n-2}0^{n-2} + \ldots. + n\,(n-1)\,(n-2)\ldots.2\cdot 1\,\Delta^{1}0^{1}$$

$$= n\cdot \lfloor\underline{n} + n\,(n-1)\,\lfloor\underline{(n-1)} + n\,(n-1)\,(n-2)\,\lfloor\underline{(n-2)} + \ldots.$$

$$+\, n\,(n-1)\,(n-2)\,\ldots.\,1\cdot \lfloor\underline{1}$$

$$= \lfloor\underline{n}\,[n + (n-1) + (n-2) + \ldots. + 1]$$

$$= \lfloor\underline{n}\,[1 + 2 + \ldots. + (n-1) + n] = \lfloor\underline{n}\,\frac{n\,(n+1)}{2}$$

$$= \frac{n\,(n+1)}{2}\cdot \Delta^n 0^n \qquad [\because \lfloor\underline{n} = \Delta^n 0^n]$$

Thus, $$\Delta^n 0^{n+1} = \frac{n\,(n+1)}{2}\,\Delta^n 0^n.$$

Example 26: ***Show that*** $\boldsymbol{(n+1)\,\Delta^n 0^n = 2\,[\Delta^{n-1} 0^n + \Delta^n 0^n].}$

Solution: By recurrence relation, we have

$$\Delta^n 0^m = n\,[\Delta^{n-1} 0^{m-1} + \Delta^n 0^{m-1}]$$

Taking $m = n + 1$, we get

$$\Delta^n 0^{n+1} = n\,[\Delta^{n+1} 0^n + \Delta^n 0^n] \qquad \ldots(1)$$

As proved in the preceding exercise, we have

$$\Delta^n 0^{n+1} = \frac{n\,(n+1)}{2}\,\Delta^n 0^n \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{n\,(n+1)}{2}\,\Delta^n 0^n = n\,[\Delta^{n-1} 0^n + \Delta^n 0^n]$$

or $$(n+1)\,\Delta^n 0^n = 2\,[\Delta^{n-1} 0^n + \Delta^n 0^n]$$

Example 27: ***Sum of the series*** $\boldsymbol{n^2 + {}^nC_1(n-1)^2 + {}^nC_2(n-2)^2 + \ldots.}$, ***n being a positive integer.***

Solution: The given series $= n^2 + {}^nC_1\,(n-1)^2 + {}^nC_2\,(n-2)^2 + \ldots.$

$$= [(x+n)^2 + {}^nC_1\,(x+n-1)^2 + {}^nC_2\,(x+n-2)^2 + \ldots.]_{x=0}$$

$$= [E^n x^2 + {}^nC_1\,E^{n-1} x^2 + {}^nC_2\,E^{n-2} x^2 + \ldots.]_{(x=0)}$$

$$= [(E^n + {}^nC_1\,E^{n-1} + {}^nC_2\,E^{n-2} + \ldots.)\,x^2]_{(x\equiv 0)}$$

$$= [(E+1)^n x^2]_{(x=0)}$$

$$= [(2+\Delta)^n x^2]_{(x=0)}$$

$$= [(2^n + {}^nC_1\,2^{n-1}\Delta + {}^nC_2\,2^{n-2}\Delta^2 + \ldots.)\,x^2]_{(x=0)}$$

$$= 2^n x^2 + {}^nC_1\,2^{n-1}\Delta\,x^2 + {}^nC_2\,2^{n-2}\Delta^2 x^2 + \ldots.]_{(x=0)}$$

$$= n\cdot 2^{n-1}\cdot 1 + \frac{n(n-1)}{1\cdot 2}2^{n-2}\cdot \lfloor\underline{2} + 0 + 0 \ldots..,$$

$$[\because \Delta^n 0^n = [\Delta^n x^n]_{(x=0)} = (n!)_{(x=0)} = \lfloor\underline{n}$$

$$\text{and } \Delta^n 0^m = [\Delta^n 0^m]_{(x=0)} = 0 \text{ for } n > m$$

$$= n\cdot 2^{n-2}\,[2 + n - 1]$$

$$= n(n+1)\,2^{n-2}$$

2.5 Separation of Symbols

For convenience we may represent the function $f(x)$ by U_x so that $f(x + nh)$ is represent by U_{x+nh} and $f(0), f(1), f(2)$, etc. may be represented by U_0, U_1, U_2 etc.

We have, $U_{x+nh} = E^n U_x = (1+\Delta)^n U_x$

$$= (1 + {}^nC_1\Delta + {}^nC_2\Delta^2 + \ldots.)\,U_x$$

$$= U_x + {}^nC_1\Delta U_x + {}^nC_2\Delta^2 U_x + \ldots.$$

This type of operation in which we separate the operand from operator is called the **separation of symbols.**

Example 28: ***Prove that*** $\boldsymbol{U_4 = U_3 + \Delta U_2 + \Delta^2 U_1 + \Delta^3 U_1}$.

Solution: R.H.S. $= U_3 + \Delta U_2 + \Delta^2 U_1 + \Delta^3 U_1$

$$= E^2 U_1 + \Delta\, E U_1 + \Delta^2\,(1+\Delta)\,U_1$$

$$= (E^2 + \Delta\, E + \Delta^2 E)\,U_1$$

$$= E\,(E + \Delta + \Delta^2)\,U_1$$

$$= E\,[E + \Delta\,(1+\Delta)]\,U_1$$

$$= E\,[E + \Delta\, E]\,U_1$$

$$= E^2\,[1+\Delta]\,U_1$$

$$= E^3 U_1 = U_4 = \text{L.H.S.}$$

Example 29: ***Show that*** $\boldsymbol{U_0 + {}^xC_1\Delta U_1 + {}^xC_2\Delta^2 U_2 + {}^xC_3\Delta^3 U_3 + \ldots.}$

$$\boldsymbol{= U_x + {}^xC_1\Delta^2 U_{x-1} + {}^xC_2\Delta^4 U_{x-2} + \ldots.}$$

Solution: R.H.S. $= U_x + {}^xC_1\Delta^2 U_{x-1} + {}^xC_2\Delta^4 U_{x-2} + \ldots.$

$$= U_x + {}^xC_1\Delta^2 E^{-1} U_x + {}^xC_2\Delta^4 E^{-2} U_x + \ldots.$$

$$= [1 + {}^xC_1\Delta^2 E^{-1} + {}^xC_2\Delta^4 E^{-2} + \ldots.]\,U_x$$

$$= [1 + \Delta^2 E^{-1}]^x\,U_x = \left[1 + \frac{\Delta^2}{E}\right]^x U_x$$

$$= E^{-x}[E + \Delta^2]^x\, U_x$$

$$= E^{-x}[E + (E - I)^2]^x\, U_x$$

$$= E^{-x}[E + E^2 + I^2 - 2\,EI]^x\, U_x$$

$$= E^{-x}[\, E^2 + I - E]^x\, U_x$$

$$= E^{-x}\,[E\,(E - I) + I]^x\, U_x$$

$$= E^{-x}[E\Delta + I]^x\, U_x \qquad [\because E - I = \Delta]$$

$$= E^{-x}[1 + \Delta\, E]^x U_x$$

$$= E^{-x}[1 + {}^xC_1\Delta\, E + {}^xC_2\ (\Delta\, E)^2 + \ldots. + {}^xC_n(\Delta\, E)^n]\, U_x$$

$$= E^{-x}U_x + {}^xC_1\Delta\, EE^{-x}U_x + {}^xC_2\,(\Delta\, E)^2\, E^{-x}U_x + \ldots.$$

$$+\, {}^xC_n(\Delta\, E)^n\, E^{-x}U_x$$

$$= U_0 + {}^xC_1\Delta U_1 + {}^xC_2\Delta^2 U_2 + \ldots.$$

$$= \text{L.H.S.}$$

Example 30: ***Prove the following identities:***

(i) $\boldsymbol{U_0 + U_1 + U_2 + \ldots. + U_n = {}^{n+1}C_1U_0 + {}^{n+1}C_2\Delta U_0 + {}^{n+1}C_3\Delta^2 U_0 + \ldots. + \Delta^n U_0}$

Solution: (i) We have, $U_0 + U_1 + U_2 + \ldots. + U_n$

$$= U_0 + EU_0 + E^2U_0 + \ldots. + E^nU_0$$

$$= (1 + E + E^2 + \ldots. + E^n)\, U_0$$

$$= \frac{E^{n+1} - 1}{E - 1}\, U_0 \qquad \text{(G.P.)}$$

$$= \frac{(1 + \Delta)^{n+1} - 1}{\Delta}\, U_0$$

$$= \frac{1}{\Delta}\,[(1 + {}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + \ldots. + \Delta^{n+1}) - 1]\, U_0$$

$$= \frac{1}{\Delta}\,[{}^{n+1}C_1\ \Delta U_0 + {}^{n+1}C_2\Delta^2 U_0 + \ldots. + \Delta^{n+1}U_0]$$

$$= {}^{n+1}C_1U_0 + {}^{n+1}C_2\Delta U_0 + {}^{n+1}C_3\Delta^2 U_0 + \ldots + \Delta^n U_0$$

(ii) $U_{x+n} = U_n + {}^xC_1\Delta U_{n-1} + {}^{x+1}C_2\Delta^2 U_{n-2} + {}^{x+2}C_3\Delta^3 U_{n-3} + \ldots.$

We have, $U_n + {}^xC_1\Delta U_{n-1} + {}^{x+1}C_2\Delta^2 U_{n-2} + {}^{x+2}C_3\Delta^2 U_{n-3} + \ldots.$

$$= [U_n + {}^xC_1\Delta\, E^{-1}U_n + {}^{x+1}C_2\Delta^2\, E^{-2}U_n + {}^{x+2}C_3\Delta^3\, E^{-3}U_n + \ldots]$$

$$= [1 + {}^xC_1\Delta\, E^{-1} + {}^{x+1}C_2\Delta^2\, E^{-2} + {}^{x+2}C_3\Delta^3\, E^{-3} + \ldots.]\, U_n$$

$$= [1 - \Delta\, E^{-1}]^{-x}U_n = \left[1 - \frac{\Delta}{E}\right]^{-x} U_n = \left[\frac{E - \Delta}{E}\right]^{-x} U_n$$

$$= \left(\frac{1}{E}\right)^{-x} U_n = E^xU_n = U_{n+x} = U_{x+n}.$$

(iii) $\Delta^n U_{x-n} = U_x - {}^nC_1 U_{x-1} + {}^nC_2 U_{x-2} - {}^nC_3 U_{x-3} + \ldots.$

We have, $U_x - {}^nC_1 U_{x-1} + {}^nC_2 U_{x-2} - {}^nC_3 U_{x-3} + \ldots.$

$$= U_x - {}^nC_1 E^{-1} U_x + {}^nC_2 E^{-2} U_x - {}^nC_3 E^{-3} U_x + \ldots.$$

$$= (1 - {}^nC_1 E^{-1} + {}^nC_2 E^{-2} - {}^nC_3 E^{-3} + \ldots.)\, U_x$$

$$= (1 - E^{-1})^n U_x = \left[1 - \frac{1}{E}\right]^n U_x = \left[\frac{E-1}{E}\right]^n U_x$$

$$= \left(\frac{\Delta}{E}\right)^n U_x = \Delta^n E^{-n} U_x = \Delta^n U_{x-n}.$$

Example 31:* *Prove the following identities:

(i) $\boldsymbol{U_x - U_{x+1} + U_{x+2} - U_{x+3} + \ldots. = \frac{1}{2}\left[U_{x-1/2} - \frac{1}{8}\Delta^2 U_{x-3/2} + \frac{1\cdot 3}{\underline{|2}}\left(\frac{1}{8}\right)^2 \Delta^4 U_{x-5/2} - \frac{1\cdot 3\cdot 5}{\underline{|3}}\left(\frac{1}{8}\right)^3 \Delta^6 U_{x-7/2} + \ldots\right]}$

Solution: We have,

$$\frac{1}{2}\left[U_{x-1/2} - \frac{1}{8}\Delta^2 U_{x-3/2} + \frac{1\cdot 3}{\underline{|2}}\left(\frac{1}{8}\right)^2 \Delta^4 U_{x-5/2} - \frac{1\cdot 3\cdot 5}{\underline{|3}}\left(\frac{1}{8}\right)^3 \Delta^6 U_{x-7/2} + \ldots\right]$$

$$= \frac{1}{2}\left[E^{-1/2}U_x - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\Delta^2 E^{-3/2} U_x + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{\underline{|2}}\left(\frac{1}{4}\right)^2 \Delta^4 E^{-5/2} U_x - \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\underline{|3}}\left(\frac{1}{4}\right)^3 \Delta^6 E^{-7/2} U_x + \ldots.\right]$$

$$= \frac{1}{2}E^{-1/2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{4}\Delta^2 E^{-1}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|2}}\left(\frac{1}{4}\Delta^2 E^{-1}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|3}}\left(\frac{1}{4}\Delta^2 E^{-1}\right)^2 + \ldots.\right]U_x$$

$$= \frac{1}{2}E^{-1/2}\left[1 + \frac{1}{4}\Delta^2 E^{-1}\right]^{-1/2} U_x$$

$$= \frac{1}{2}E^{-1/2}\left[\frac{4E + \Delta^2}{4E}\right]^{-1/2} U_x = \frac{1}{2}E^{-1/2}\left[\frac{4 + 4\Delta + \Delta^2}{4E}\right]^{-1/2} U_x$$

$$= \frac{1}{2} E^{-1/2} \cdot 2E^{1/2} \{(2+\Delta)^2\}^{-1/2} U_x = (2+\Delta)^{-1} U_x$$

$$= U_x (1+E)^{-1} = [1 - E + E^2 - E^3 + E^4 - E^5 + \ldots.] U_x$$

$$= U_x - U_{x+1} + U_{x+2} - U_{x+3} + \ldots.$$

(ii) $U_0 + {}^nC_1 U_1 x + {}^nC_2 U_2 x^2 + {}^nC_3 U_3 x^3 + \ldots.$

$$= (1+x)^n U_0 + {}^nC_1 (1+x)^{n-1} x\Delta U_0 + {}^nC_2 (1+x)^{n-2} x^2 \Delta^2 U_0 + \ldots.$$

Solution: We have,

$$(1+x)^n U_0 + {}^nC_1 (1+x)^{n-1} x\, \Delta U_0 + {}^nC_2 (1+x)^{n-2} x^2 \Delta^2 U_0 + \ldots.$$

$$= \{(1+x) + x\Delta\}^n U_0 = \{1 + x(1+\Delta)\}^n U_0$$

$$= (1+xE)^n U_0 = [1 + {}^nC_1 xE + {}^nC_2 x^2 E^2 + {}^nC_3 x^3 E^3 + \ldots.] U_0$$

$$= U_0 + {}^nC_1 x U_1 + {}^nC_2 x^2 U_2 + {}^nC_3 x^3 U_3 + \ldots$$

Example 32: ***Show that***

$$\Delta x^m - \frac{1}{2}\Delta^2 x^m + \frac{1\cdot 3}{2\cdot 4}\Delta^3 x^m - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Delta^4 x^m + \ldots.\ m \text{ terms}$$

$$= \left(x+\frac{1}{2}\right)^m - \left(x-\frac{1}{2}\right)^m.$$

Solution: We know that $\Delta^k x^m = 0$ for $k > m$; thus in this case the sum of m terms will be same as the sum of infinite terms.

Thus, we have L.H.S. $= \Delta x^m - \frac{1}{2}\Delta^2 x^m + \frac{1\cdot 3}{2\cdot 4}\Delta^3 x^m - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Delta^4 x^m + \ldots.\ m$ terms

$$= \Delta\left[1 - \frac{\Delta}{2} + \frac{1\cdot 3}{2\cdot 4}\Delta^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Delta^3 + \ldots.\ m \text{ terms}\right] x^m$$

$$= \Delta\left[1 - \frac{1}{2}\Delta + \frac{1\cdot 3}{2\cdot 4}\Delta^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Delta^3 + \ldots.\ \text{to } \infty\right] x^m$$

$$= \Delta\,[1+\Delta]^{-1/2} x^m = \Delta E^{-1/2} x^m$$

$$= \Delta\left(x - \frac{1}{2}\right)^m \qquad \because \quad h = 1$$

$$= \left(x+\frac{1}{2}\right)^m - \left(x-\frac{1}{2}\right)^m = \text{R.H.S.}$$

Example 33: ***Sum to n terms***

$$\mathbf{1\cdot 2\,\Delta x^n - 2\cdot 3\Delta^2 x^n + 3\cdot 4\Delta^3 x^n - 4\cdot 5\Delta^4 x^n + \ldots}$$

Solution: Here the nth term $= (-1)^{n-1} n(n+1)\,\Delta^n x^n$.

We know that $\Delta^k x^n = 0$ for $k > n$

Thus, in this case sum of n terms will be same as sum to infinity

$$\therefore \quad \text{required sum} = (1\cdot 2\,\Delta - 2\cdot 3\,\Delta^2 + 3\cdot 4\,\Delta^3 - 4\cdot 5\,\Delta^4 + \dots \infty]\, x^n$$

$$= 2\,\Delta\,(1 - 3\,\Delta + 6\,\Delta^2 - 10\Delta^3 + \dots)\, x^n$$

$$= 2\,\Delta\,(1+\Delta)^{-3}\, x^n$$

$$= 2\,\Delta E^{-3} x^n$$

$$= 2\,\Delta\,(x-3)^n = 2\,\{(x-2)^n - (x-3)^n\}, h = 1$$

Problem Set

1. Evaluate:
   (i) $\Delta^3 [ax^3 + bx^2 + cx + d]$ **[B.C.A. (Agra) 2004]**
   (ii) $\Delta \tan ax$
   (iii) $\Delta \tan^{-1} ax$

2. (i) $\Delta^3 (1-x)(1-2x)(1-3x)$ **[B.C.A. (Rohilkhand) 2008, 2010]**
   (ii) $\Delta^n (e^{ax+b})$

3. Evaluate $\Delta^4 ae^x$

4. $\Delta^2 \left[\dfrac{a^{2x} + a^{4x}}{(a^2-1)^2}\right]$.

5. $\Delta^n \cos(a + bx)$ **[B.C.A. (Avadh) 2009]**

6. For the function $f(x) = c^x$ then find $\Delta^n c^x$.

7. What is the difference between $\left(\dfrac{\Delta U_x}{EU_x}\right)^2$ and $\left(\dfrac{\Delta^2 U_x}{E^2 U_x}\right)$. If $U_x = x^3$ and interval of differencing is one, find expression for both.

8. Find the function where first difference is
   (i) a (ii) $ax + b$ (iii) $\sin x$ (iv) e^{a+bx} (v) 5^x.
   [B.B.A. (Meerut) 2011]

9. If the interval of differencing is unity, then show that $x^{(n)} = \underline{\lfloor n}\; {}^xC_n$. Hence or other will prove that
   (i) $\Delta^r\; {}^xC_n = {}^xC_{n-r}$ (ii) $\Delta^n\; {}^xC_n = 1$

10. Find the value of

(i) $\frac{\Delta}{\Delta x}[3x^{(4)} - 2x^{(3)} - 3x^{(2)} + x - 5]$ **[B.B.A. (Meerut) 2008]**

(ii) $\Delta[2x^{(-2)} - 3x^{(3)} - 4x^{(-1)}]$

(iii) $\frac{\Delta^2}{\Delta x^2}[3x^{(4)} - 5x^{(-2)}]$ **[B.B.A. (Meerut) 2009]**

11. Show that $x^{(r)}(x - rh)^{(n)} = x^{(r+n)}$.

12. Express the function $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$ in factorial notation, the interval of difference being unity. Find $\Delta^4 f(x)$.

13. Obtain a function whose first difference is

$$x^3 + 2x^2 + 5x + 12$$

[B.B.A. (Meerut) 2010]

14. Prove that

$$\frac{\Delta^n 0^m}{\lfloor n} = \frac{n\Delta^n 0^{m-1}}{\lfloor n} + \frac{\Delta^{n-1} 0^{m-1}}{\lfloor n-1}.$$

15. Sum of the series

$$(n-1)^2\ {}^nC_1 + (n-3)^2\ {}^nC_3 + (n-5)^2\ {}^nC_5 + \ldots.$$

16. Show that

$U_4 = U_0 + 4\Delta U_0 + 6\Delta^2 U_{-1} + 10\Delta^3 U_{-1}$ if fourth and higher difference are zero.

17. Find (i) $U_0 + \frac{U_1 x}{\lfloor 1} + \frac{U_2 x^2}{\lfloor 2} + \frac{U_3 x^3}{\lfloor 3} + \ldots.$

(ii) $U_x - \frac{1}{8}\Delta^2 U_{x-1} + \frac{1\cdot 3}{8\cdot 16}\Delta^4 U_{x-2} - \frac{1\cdot 3\cdot 5}{8\cdot 16\cdot 24}\Delta^6 U_{x-3} + \ldots$

(iii) $U_{2n} - {}^nC_1 2\, U_{2n-1} + {}^nC_2 2^2 U_{2n-2} + \ldots. + (-2)^n U_n$

18. Prove that

$$\sum_{x=1}^{\infty} U_{2x} = \frac{1}{2}\sum_{x=0}^{\infty} U_x + \frac{1}{4}\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \ldots.\right]U_0$$

19. If $\Delta^3 U_x = 0$, prove that

$$U_{x+1/2} = \frac{1}{2}[U_x + U_{x+1}] - \frac{1}{16}[\Delta^2 U_{x+1} + \Delta^2 U_x]$$

Answers

1.	(i) $6ah^3$ (ii) $\dfrac{\sin a}{\cos ax \cos a(x+1)}$ (iii) $\tan^{-1}\dfrac{a}{1+a^2x^2+a^2x}$
2.	(i) -36 (ii) $e^{ax+b}(e^a-1)^n$
3.	$a(e-1)^4 e^x$
4.	$a^{2x}+(a^2+1)^2 a^{4x}$
5.	$\left(2\sin\dfrac{b}{2}\right)^n \cos\left[\left\{a+bx+n\left(\dfrac{\pi+b}{2}\right)\right\}\right]$
6.	$c^x \sum\limits_{r=0}^{n}\dbinom{n}{r}(-1)^{n-r}c^{rh}$
7.	$\dfrac{6(x+1)}{x^3+6x^2+12x+8}$
8.	(i) $F(x)=ax+b$ (ii) $F(x)=\dfrac{1}{2}ax^2+\left(b-\dfrac{1}{2}a\right)x+c$ (iii) $F(x)=-\dfrac{1}{2}\sin x$ (iv) $F(x)=\dfrac{e^{a+bx}}{e^b-1}$ (v) $F(x)=\dfrac{1}{4}5^x$
10.	(i) $12x^{(3)}-6x^{(2)}-6x^{(1)}+1$ (ii) $[-4x^{(-3)}-9x^{(2)}+4x^{(-2)}]\,h$ (iii) $36x^{(2)}-30x^{(-4)}$
12.	24
13.	$\dfrac{x^{(4)}}{4}+\dfrac{5x^{(3)}}{3}+4x^{(2)}+12x^{(1)}+k$
15.	$\dfrac{1}{2}[n2^{n-1}+n(n-1)\,2^{n-2}]$
17.	(i) $e^x\left[U_0+x\Delta U_0+\dfrac{x^2}{2!}\Delta^2U_0+\dfrac{x^3}{3!}\Delta^3U_0+\ldots.\right]$ (ii) $U_{x+1/2}-\dfrac{1}{2}\Delta U_{x+1/2}+\dfrac{1}{4}\Delta^2U_{x+1/2}\ \ldots.$ (iii) $(-1)^n(c-2an)$ where $U_n=an^2+bn+c$

○○○

Unit-2

Chapter 3

Interpolation with Equal Intervals

3.1 Interpolation

Interpolation is an operation of estimating the values of a function for any intermediate value of the argument when the values of the function corresponding to a number of the value of the argument are given. Extrapolation is the method of estimating the values for the past period or future period from the given values of argument.

3.1.1 Assumptions for Interpolation

There are two main assumptions made for interpolation or extrapolation from the given set of values of the argument and corresponding values of function.

1. The function is assumed to be expressible in the form of polynomial of the variables.
2. There are no sudden jumps or falls in the value of the function from one period to another.

3.1.2 Methods for Interpolation

1. Graphical Method
2. Algebraic Method.

   But we shall consider only the method based on finite differences.

3.1.3 Newton's Binomial Expansion Method for Interpolation or Newton's Interpolation Formula

Let we have a set of equidistant values of x and corresponding values of $f(x)$, then we have to find the missing terms of the independent variable x.

Let $x = a, a + h, a + 2h, \dots, a + nh$ be $(n + 1)$ equidistant values.

Then corresponding functions as

$$f(a), f(a+h), f(a+2h) \dots . f(a+nh)$$

Let one of the values of $f(x)$ say $f(i)$ be missing. We have to interpolate this missing value of $f(x)$. Actually we are given n values of $f(x)$ corresponding to n values of x. Let us assume that the function $f(x)$ can be represented by a polynomial.

Since n values of $f(x)$ are known then we can consider that $f(x)$ be a polynomial of degree $(n-1)$.

Hence, $\Delta^{n-1} f(x) = \text{constant} \Rightarrow \Delta^n f(x) = 0$

or $\quad (E-I)^n f(x) = 0 \quad \therefore \quad \Delta = E - I$

or $\quad [E^n - {}^nC_1 E^{n-1} I + {}^nC_2 E^{n-2} I \dots . (-1)^n]\, f(x) = 0$

or $\quad E^n f(x) - {}^nC_1 E^{n-1} f(x) + {}^nC_2 E^{n-2} f(x) + \dots + (-1)^n f(x) = 0 \quad \dots(1)$

In this equation each term except missing term is known and hence we can find the missing term. If two term are missing when we used $f(x)$ can be represented by a polynomial of $(n-2)$ degree so that

$$\Delta^{n-1} f(x) = 0 \quad i.e., \ (E-I)^{n-1} f(x) = 0$$

or $\quad [E^{n-1} - {}^{n-1}C_1 E^{n-2} I + {}^{n-1}C_2 E^{n-3} I^2 \dots (-1)^{n-1} I^{n-1}]\, f(x) = 0$

This gives for $x = 0$ (first tabular value)

$$f(n-1) - {}^{n-1}C_1 f(n-2) + {}^{n-2}C_2 f(n-3) \dots + (-1)^{n-1} f(0) = 0 \quad \dots(2)$$

and for $x = 1$ (second tabulated value)

$$f(n) - {}^{n-1}C_1 f(n-1) + {}^{(n-1)}C_2 f(n-2) + \dots + (-1)^{n-1} = 0 \quad \dots(3)$$

Solving these two equations (2) and (3), we can find the two missing values.

Similarly, if three values are missing we will need three equations and so on.

SOLVED EXAMPLES

Example 1: ***Evaluate the missing term in the following:***

x:	100	101	102	103	104
$f(x) = \log x$:	2.000	2.0043	–	2.0128	2.0170

Solution: As four values of $f(x)$ are known so

$$\Delta^4 f(x) = 0 \quad \text{or} \quad [E-I]^4 f(x) = 0$$

$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$$

$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

For $x = 100$, we have

$$f(104) - 4f(103) + 6f(102) - 4f(101) + f(100) = 0$$

Substituting the known values, we get

$$2.0170 - 4 \times 2.0128 + 6f(102) - 4 \times 2.0043 + 2 = 0$$

$$6f(102) = 16.0684 - 4.0170 = 12.0514$$

or giving $f(102) = 2.0086$ *i.e.*, $\log 102 = 2.0086$.

Example 2: ***Estimate the missing term in the following:***

x	1	2	3	4	5	6	7
y	2	4	8	?	32	64	128

[B.B.A. (Meerut) 2008, 2010]

Solution: Here, six values of x and y are given, therefore, we can assume the given function expressible as a polynomial of degree five.

Hence, $\Delta^6 y = 0$ *i.e.*, $\Delta^6 f(x) = 0$

or $(E - I)^6 f(x) = 0$

or $[E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 + {}^6C_4 E^2 - {}^6C_5 E + {}^6C_6] f(x) = 0$

or $E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6Ef(x) + f(x) = 0$

or $f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0$

For $x = 1$, we have

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

or $$128 - 6 \times 64 + 15 \times 32 - 20f(4) + 15 \times 8 - 6 \times 4 + 2 = 0$$

$$128 - 384 + 480 - 20f(4) + 120 - 24 + 2 = 0$$

$$20f(4) = 730 - 408$$

$$= 322$$

$\therefore$ $$f(4) = 16.1$$

Example 3: ***Obtain the estimate of the missing figure in the following table:***

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	?	64	?	216	343	512

[B.C.A. (Rohilkhand) 2006, 2011]

Solution: As six values of $[x, f(x)]$ are given therefore we may assume the function to be represented by a polynomial of degree five.

So that, $\Delta^5 f(x) = \text{constant}$ and $\Delta^6 f(x) = 0$

$$[E-1]^6 f(x) = 0$$

$$[E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 + {}^6C_2 E^2 - {}^6C_1 E + 1]\ f(x) = 0$$

$$f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f\ (x+1) + f(x) = 0 \quad \ldots(1)$$

For $x = 1$, (1) becomes

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

Putting the known values, we get

$$343 - 6 \times 216 + 15f(5) - 20 \times 64 + 15f(3) - 6 \times 8 + 1 = 0$$

$$15f(5) + 15f(3) = 2280$$

$$f(5) + f(3) = 152 \quad \ldots(2)$$

For $x = 2$, (1) becomes

$$f(8) - 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0$$

$$512 - 6 \times 343 + 15 \times 216 - 20f(5) + 15 \times 64 - 6f(3) + 8 = 0$$

$$20f(5) + 6f(3) = 2662$$

$$10f(5) + 3f(3) = 1331 \quad \ldots(3)$$

Solving (2) and (3), we get

$$f(3) = 27 \quad \text{and} \quad f(5) = 125$$

Example 4: ***Find $f(6)$; it is given $f(0) = -3$, $f(1) = 6$, $f(2) = 8$, $f(3) = 12$, the third difference being constant.***

Solution: The difference table for given values is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	– 3			
		9		
1	6		– 7	
		2		9
2	8		2	
		4		
3	12			

Now $f(6) = E^6 f(0) \quad \because$ x starts from $x = 0$ and $h = 1$

$$= (1+\Delta)^6 f(0)$$

$$= [1 + {}^6C_1\Delta + {}^6C_2\Delta^2 + {}^6C_3\Delta^3 + {}^6C_4\Delta^4 + {}^6C_5\Delta^5 + {}^6C_6\Delta^6]\ f(0)$$

$$= f(0) + 6\Delta f(0) + 15\Delta^2 f(0) + 20\Delta^3 f(0) + 0 + 0 + 0$$

$$= -3 + 6 \times 9 + 15 \times (-7) + 20 \times 9$$

$$= -3 + 54 - 105 + 180 = 126$$

Example 5: ***Estimate the production of cotton in the year 1935 from the data given below:***

Year (x) :	1931	1932	1933	1934	1935	1936	1937
Production $f(x)$: **(in millions)**	17.1	13.0	14.0	9.6	–	12.4	18.2

[B.C.A. (Kanpur) 2010]

Solution: As six observations are given, therefore we can fit a polynomial of fifth degree and hence the sixth difference will be equal to zero.

$$\therefore \quad \Delta^6 f(x) = 0$$

$$(E - I)^6 f(x) = 0$$

$$[E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 + {}^6C_4 E^2 - {}^6C_5 E + {}^6C_6]\, f(x) = 0$$

$$E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6Ef(x) + f(x) = 0$$

$$f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0$$

Taking $x = 1931$, we get

$$f(1937) - 6f(1936) + 15f(1935) - 20f(1934) + 15f(1933) - 6f(1932) + f(1931) = 0$$

$$18.2 - 6 \times 12.4 + 15 \times f(1935) - 20 \times 9.6 + 15 \times 14 - 6 \times 13 + 17.1 = 0$$

$$15f(1935) = 99.1 \text{ giving } f(1935) = 6.60$$

Example 6: ***From the following data find the value of U_{47}:***

$$U_{46} = 0.2884, U_{48} = 0.5356, U_{49} = 0.6513 \text{ and } U_{50} = 0.7620.$$

Solution: Since four values are given, we have

$$\Delta^4 U_x = 0 \quad \text{or} \quad (E - I)^4 U_x = 0$$

$$[E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + I]\, U_x = 0$$

$$U_{x+4} - 4U_{x+3} + 6U_{x+2} - 4U_{x+1} + U_x = 0$$

for $x = 46$, we have

$$U_{50} - 4U_{49} + 6U_{48} - 4U_{47} + U_{46} = 0$$

$$0.7620 - 4 \times 0.6513 + 6 \times 0.5356 - 4U_{47} + 0.2884 = 0$$

$$0.7620 - 2.6052 + 3.2136 - 4U_{47} + 0.2884 = 0$$

$$4U_{47} = 1.6588 \quad \text{or} \quad U_{47} = 0.4147$$

3.2 Newton-Gregory Forward Difference Interpolation Formula

If $a, a+h, a+2h, \ldots, a+nh$ are $(n+1)$ equi-spaced values of argument x so that a function $y = f(x)$ assume the values $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$, then

$$f(a+mh) = f(a) + \frac{m^{(1)}}{1!}\Delta f(a) + \frac{m^{(2)}}{2!}\Delta^2 f(a) + \ldots + \frac{m^{(n)}}{n!}\Delta^n f(a)$$

$$= \sum_{r=0}^{n} \frac{m^{(r)}}{r!}\Delta^r f(a), \text{where } m^{(r)} = m(m-1)\ldots.\{m-(r-1)\}.$$

Proof: We assume that the given function $f(x)$ can be expressed as a polynomial of degree n in x representing this in factorial notations. Let

$$f(x) = A_0 + A_1(x-a)^{(1)} + A_2(x-a)^{(2)} + \ldots + A_n(x-a)^{(n)}$$

$$= A_0 + A_1(x-a) + A_2(x-a)\{x-(a+h)\} + \ldots.$$

$$+ A_n(x-a)\{x-(a+h)\} \ldots. [x - \{a+(n-1)h\}] \qquad \ldots(1)$$

where $A_0, A_1, A_2, \ldots., A_n$ are constants.

To find these constants, put $x = a, a+h, a+2h, \ldots., a+nh$ in (1).

Thus, for $\quad x = a, (1) \Rightarrow f(a) = A_0 \Rightarrow A_0 = f(a),$

and for $\quad x = a+h, (1) \Rightarrow f(a+h) = A_0 + hA_1$

$$\Rightarrow \quad A_1 = \frac{f(a+h) - f(a)}{h} = \frac{\Delta f(a)}{h}$$

and for $\quad x = a+2h, (1) \Rightarrow f(a+2h) = A_0 + 2hA_1 + 2h \cdot h \cdot A_2$

$$\Rightarrow \quad A_2 = \frac{[f(a+2h) - 2\{f(a+h) - f(a)\} + f(a)]}{2h^2}$$

$$= \frac{\Delta^2 f(a)}{2h^2} = \frac{\Delta^2 f(a)}{2!\,h^2}$$

and so on. Thus, $A_n = \frac{1}{n!\,h^n}\Delta^n f(a)$

Substituting these values of $A_0, A_1, A_2, \ldots., A_n$ in (1), we get

$$f(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!\,h^2}(x-a)\{x-(a+h)\} + \ldots.$$

$$+ \frac{\Delta^n f(a)}{n!\,h^n}(x-a)\{x-(a+h)\} \ldots. [x - \{a+(n-1)h\}]$$

Putting $\quad \frac{x-a}{h} = m$ (say) so that $x = a + mh,$ we have

$$f(a+mh) = f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \ldots. + \frac{m(m-1) \ldots. m-(n-1)}{n!}\Delta^n f(a)$$

$$f(a+mh) = f(a) + m^{(1)}\Delta f(a) = \frac{m^{(2)}}{2!}\Delta^2 f(a) + \ldots. + \frac{m^{(n)}}{n!}\Delta^n f(a)$$

This is Newton Gregory formula for forward interpolation. This is generally used for interpolating the values of $f(x)$, when x is slightly greater or slightly less than 0.

Aliter: We know that

$$f(a+xh) = E^x f(a) = (1+\Delta)^x f(a)$$

$$f(a+xh) = [1 + {}^xC_1\Delta + {}^xC_2\Delta^2 + \ldots. + {}^xC_n\Delta^n + \ldots.]\, f(a)$$

$$= f(a) + {}^xC_1\Delta f(a) + {}^xC_2\Delta^2 f(a) + \ldots. + {}^xC_n\Delta^n f(a)$$

as we have assumed $f(x)$ to be a polynomial of nth degree so $\Delta^{n+1} f(a)$, etc. will be equal to zero

$$f(a+xh) = f(a) + x^{(1)}\Delta f(a) + \frac{x^{(2)}}{2!}\Delta^2 f(a) + \ldots . + \frac{x^{(n)}}{n!}\Delta^n f(a)$$

which is the required formula.

Cor: Newton-Gregory Advancing Difference Formula

If we take $a = 0, h = 1$ and replace m by x, then the Newton Gregory forward difference formula become

$$f(x) = f(0) + {}^xC_1\Delta f(0) + {}^xC_2\Delta^2 f(0) + \ldots . + {}^xC_n\Delta^n f(0).$$

3.3 Newton-Gregory Backward Difference Interpolation Formula

This formula is used to interpolate the value of $f(x)$ when x is much greater than a *i.e.*, if the value to be interpolated lies on the other side.

Let $y = f(x)$ be the given function. Assuming that $f(x)$ can be represented by a polynomial of nth degree in x, and expressing it in factorial notation, we suppose

$$f(x) = A_0 + A_1[x-(a+nh)]^{(1)} + A_2[x-(a+nh)]^{(2)} + \ldots . + A_n[x-(a+nh)]^{(n)} \quad \ldots(1)$$

Where $A_0, A_1, A_2 \ldots . A_n$ are constants

or
$$f(x) = A_0 + A_1\,[x-(a+nh)] + A_2\,[x-(a+nh)]$$
$$\times[x-\{a+(n-1)h\}] + \ldots . + A_n[x-(a+nh)] \ldots . [x-(a+h)]$$

Putting
$$x = a+nh \quad \Rightarrow \quad f(a+nh) = A_0 \ \text{ or } \ A_0 = f(a+nh)$$
$$x = a+(n-1)\,h \ \Rightarrow \ f\{a+(n-1)h\} = A_0 + A_1(-h)$$

or
$$A_1 = \frac{f(a+nh) - f(a+nh-h)}{h} \ \text{ or } \ A_1 = \frac{\nabla f(a+nh)}{h}$$
$$x = a+(n-2)\,h \ \Rightarrow \ f\{a+(n-2)h\} = A_0 + A_1(-2h) + A_2(-2h)(-h)$$

or
$$A_2 = \frac{f(a+nh) - 2f\,\{a+(n-1)h\} + f\{a+(n-2)\,h\}}{2h^2}$$
$$= \frac{\nabla^2 f(a+nh)}{\underline{|2}\,h^2}$$

and so on. Thus, $A_n = \dfrac{\nabla^n f(a+nh)}{\underline{|n}\,h^n}$

Substituting the value of $A_0, A_1, A_2, \ldots . A_n$ in (1), we get

$$f(x) = f(a+nh) + \frac{\nabla f(a+nh)}{h}[x-(a+nh)] + \ldots .$$
$$+ \frac{\nabla^n f(a+nh)}{\underline{|n}\,h^n}\,[x-(a+nh)]\,[x-\{a+(n-1)\,n\}] \ldots [x-(a+h)] \quad \ldots(2)$$

Let $\dfrac{x-(a+nh)}{h}=m$

or $x=a+nh+mh$ then,

$$f(a+nh+mh)=f(a+nh)+m\nabla f(a+nh)+\frac{m\,(m+1)}{\lfloor\underline{2}}\times\nabla^2 f(a+nh)+\ldots.$$

$$+\frac{m\,(m+1)\ldots.(m+n-1)}{\lfloor\underline{n}}\,\nabla^n f(a+nh) \qquad \ldots(3)$$

This is the form in which Newton-Gregory formula for backward interpolation is often written.

3.3.1 To Express Newton's Backward Difference Formula in Terms of Forward Difference

By definition, we have

$$\nabla f(a+nh)=f(a+nh)-f(a+\overline{n-1}\,h)$$

and $\Delta\, f\{a+(n-1)\,h\}=f(a+nh)-f\{a+(n-1\,h)\}$

Therefore $\nabla f(a+nh)=\Delta\, f\{a+(n-1)\,\mathrm{h}\}$

Similarly $\nabla^2 f(a+nh)=\Delta^2 f\{a+(n-2)\,h\}$ $\qquad\ldots(1)$

. . . .

$$\nabla^n f(a+nh)=\Delta^n f\{a+(n-n)\,h\}=\Delta^n f(a)$$

Using these in (2), we have

$$f(x)=f(a+nh)+\{x-(a+nh)\}\,\frac{\Delta f\{a+(n-1)\,h\}}{h}$$

$$+\{x-(a+nh)\}\,[x-\{a+(n-1)\,h\}]\cdot\frac{\Delta^2}{\lfloor\underline{2}\,h^2}\,f\{a+(n-2)\,h\}+\ldots.$$

$$+\{x-(a+nh)\}\ldots.\{x-a\}\cdot\frac{\Delta^n}{\lfloor\underline{n}\,h^n}\,f(a).$$

Now let $x=a, a+h, a+2h,\ldots, a+nh$ be denoted by

$$x_0, x_1, x_2,\ldots.\, x_n$$

and the corresponding value of $f(x)$ by $y_0, y_1, y_2,\ldots..\, y_n$ then the above formula can be written as

$$f(x)=y_n+(x-x_n)\,\frac{\Delta}{h}\,y_{n-1}+(x-x_n)\,(x-x_{n-1})\,\frac{\Delta^2}{\lfloor\underline{2}\,h^2}\,y_{n-2}+\ldots$$

$$+(x-x_n)\,(x-x_{n-1})\ldots.(x-x_0)\,\frac{\Delta^n}{\lfloor\underline{n}\,h^n}\,y_0$$

giving the Newton's backward difference formula in terms of forward differences.

3.3.2 Prove the Backward Difference Formula for Interpolation

$$f(x)=f(0)+x\nabla f(0)+\frac{x\,(x+1)}{\lfloor\underline{2}}\,\nabla^2 f(0)+\ldots.+\frac{x\,(x+1)\ldots.(x+n+1)}{\lfloor\underline{n}}\,\nabla^n f(0).$$

Proof: By definition, we have

$$E=(1-\nabla)^{-1}$$

$$\therefore \qquad f(x) = E^x f(0) = (1-\nabla)^{-x} f(0)$$

$$f(x) = \left[1 + x\nabla + \frac{(-x)\,(-x-1)}{\underline{|2}}\nabla^2 + \ldots + \frac{(-x)\ldots\ldots\{-x-(n-1)\}}{\underline{|n}}\nabla^n\right] f(0)$$

$$= f(0) + x\,\nabla f(0) + \frac{x(x+1)}{\underline{|2}}\nabla^2 f(0) + \ldots + \frac{x\,(x+1)\ldots(x+n-1)}{\underline{|n}}\nabla^n f(0)$$

$$f(x) = f(0) + x\Delta\, f(-1) + \frac{x\,(x+1)}{\underline{|2}}\Delta^2 f(-2) + \ldots + \frac{x\,(x+1)\ldots(x+n-1)}{\underline{|n}}\Delta^n f(-n)$$

Example 7: ***Given below are the figures of population of a district for different years. Find the population for 1975:***

Year :	1951	1961	1971	1981	1991
Population (In Lakhs) :	7	9	6	14	16

[B.B.A. (Meerut) 2008]

Solution: The difference table is as given below:

Year x	**Population $f(x)$**	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1951	7				
		2			
1961	9		–5		
		–3		16	
1971	6		11		–33
		8		–17	
1981	14		–6		
		2			
1991	16				

We want $f(1975) = f(a+uh)$, where $a = 1951, h = 10$

$\therefore \qquad a + uh = 1975 \quad \Rightarrow \; 1951 + u \times 10 = 1975$

$\Rightarrow \qquad 10\,u = 24$

$\Rightarrow \qquad u = \dfrac{24}{10} = 2.4$

By Newton's formula for forward interpolation, we get

$$f(a+uh) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{\underline{|2}}\Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{\underline{|4}}\Delta^4 f(a)$$

$$f(1975) = f(1951) + 2.4 \times \Delta f(1951) + \frac{2.4(2.4-1)}{\lfloor 2} \Delta^2 f(1951)$$

$$+ \frac{(2.4)(2.4-1)(2.4-2)}{\lfloor 3} \Delta^3 f(1951)$$

$$+ \frac{(2.4)(2.4-1)(2.4-2)(2.4-3)}{\lfloor 4} \Delta^4 f(1951)$$

$$= 7 + 2.4 \times 2 + \frac{2.4 \times 1.4}{2} \times (-5) + \frac{(2.4)(1.4)(0.4)}{6} \times 16$$

$$+ \frac{(2.4)(1.4)(0.4)(-0.6)}{24} \times (-33)$$

$$= 7 + 4.8 - 8.4 + 3.584 + 1.1088$$

$$= 8.0928$$

Example 8: ***The following table is given:***

x :	0	1	2	3	4
$f(x)$:	3	6	11	18	27

What is the form of the function $f(x)$?

Solution: The table of the finite difference is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

By Newton-Gregory interpolation formula, we have

$$f(a+mh) = f(a) + m\Delta f(a) + \frac{m(m-1)}{\lfloor 2} \Delta^2 f(a) + \ldots$$

Here, $a = 0, h = 1;\quad \therefore\quad m = \frac{x-a}{h} = \frac{x-0}{1} = x.$

Thus, $f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{\lfloor 2} \Delta^2 f(0)$

Putting values from the table, we have

$$f(x) = 3 + x \times 3 + \frac{x\,(x-1)}{2} \times 2$$

$$f(x) = 3 + 3x + x^2 - x = x^2 + 2x + 3$$

Example 9: ***From the following table of yearly premiums for policies maturing at quinquennial ages, estimate the premium for policies maturing at the age of 46 years*:**

Age x :	45	50	55	60	65
Premium y_x :	2.871	2.404	2.083	1.862	1.712

Solution: The table of finite difference is given below:

x	$y_x = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
45	2.871				
		–.467			
50	2.404		.146		
		–.321		–.046	
55	2.083		.100		– 0.017
		–.221		–.029	
60	1.862		.071		
		–.150			
65	1.712				

Newton-Gregory forward interpolation formula is

$$f(a+mh) = f(a) + {}^{m}C_1 \Delta f(a) + {}^{m}C_2 \Delta^2 f(a) + \dots \qquad \dots(1)$$

Here we have to interpolate $f(46)$

$$\therefore \qquad a + mh = 46;\ m = \frac{46-45}{5} = \frac{1}{5} = 0.2,\ a = 45$$

$\therefore$ From (1), we get

$$f(46) = f(45) + \frac{1}{5}\Delta f(45) + \frac{1}{5}\left(\frac{-4}{5}\right)\frac{1}{\lfloor\underline{2}}\Delta^2 f(45)$$

$$+ \frac{1}{5}\left(\frac{-4}{5}\right)\left(\frac{-9}{5}\right)\frac{1}{\lfloor\underline{3}}\Delta^3 f(45) + \frac{1}{5}\left(\frac{-4}{5}\right)\left(\frac{-9}{5}\right)\left(\frac{-14}{5}\right)\frac{1}{\lfloor\underline{4}}\Delta^4 f(45)$$

$$f(46) = 2.871 - 0.467 \times 0.2 + \frac{0.146}{2} \times 0.2\,(-0.8)$$

$$+ \frac{-0.046 \times 0.2 \times (-0.8) \times (-1.8)}{3 \times 2 \times 1} + \frac{0.017 \times 0.2 \times (-0.8) \times (-1.8) \times (-2.8)}{4 \times 3 \times 2 \times 1}$$

$$= 2.871 - 0.0934 - 0.01168 - 0.002208 - 0.0005712$$

$$f(46) = 2.763$$

Example 10: ***Estimate the sale for 1966 using the following table:***

Year :	1931	1941	1951	1961	1971	1981
Sale in (1000) :	12	15	20	27	39	52

Solution: Let us introduce a new variable m given by $m = \dfrac{x - 1931}{10}$,

$\therefore$ m takes the values 0, 1, 2, 3, 4.

The forward difference table is as order:

x	X	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1931	0	12					
			3				
1941	1	15		2			
			5		0		
1951	2	20		2		3	
			7		3		–10
1961	3	27		5		–7	
			12		– 4		
1971	4	39		1			
			13				
1981	5	52					

By Newton-Gregory formula

$$f(x) = f(0) + m\Delta f(0) + \frac{m(m-1)}{\lfloor 2}\Delta^2 f(0) + \frac{m(m-1)(m-2)}{\lfloor 3}\Delta^3 f(0) + \ldots$$

Here $$m = \frac{x-a}{h} = \frac{1966 - 1931}{10} = \frac{35}{10} = 3.5$$

Thus, $$f(1966) = 12 + 3.5 \times 3 + \frac{35 \times 2.5}{2} \times 2 + \frac{3.5 \times 2.5 \times 1.5}{6} \times 0$$

$$+ \frac{3.5 \times 2.5 \times 1.5 \times 0.5}{24} \times 3 + \frac{3.5 \times 2.5 \times 1.5 \times 0.5 \times (-0.5)}{120} \times (-10)$$

$$= 12 + 10.5 + 87.5 + 0.82031 + 0.2734$$

$$= 111.09371$$

Example 11: ***If l_x represents the numbers living at age x in a life table, find accurately as data will permit, l_x for value of $x = 35, 42, 47$ given $I_{20} = 512, l_{30} = 439, l_{40} = 346$ and $I_{50} = 243$.***

Solution: For the given values the difference table is as follows:

x	$l_x = f(x)$	ΔI_x	$\Delta^2 l_x$	$\Delta^3 l_x$
20	512			
		– 73		
30	439		– 20	
		– 93		10
40	346		– 10	
		– 103		
50	243			

By Newton Gregory interpolation formula,

$$f(a+mh) = f(a) + m\Delta f(a) + \frac{m(m-1)}{\underline{|2}}\Delta^2 f(a) + \frac{m(m-1)(m-2)}{\underline{|3}}\Delta^3 f(a) + \ldots \quad \ldots(1)$$

1. To interpolate l_{35}: Here $a = 20, h = 10$; for $x = 35$, we have

$$m = \frac{x-a}{h} = \frac{35-20}{10} = 1.5$$

From (1)

$$I_{35} = 512 + 1.5 \times (-73) + \frac{1.5 \times 0.5}{2} \times (-20) + \frac{1.5 \times 0.5\,(-0.5)}{6} \times 10$$

$$= 512 - 109.5 - 7.5 - 0.625 = 394.375$$

$$\simeq 394$$

2. To interpolate l_{42}:

Here $a = 20, h = 10$ and $m = \frac{42-20}{10} = 2.2$

∴ From (1)

$$l_{42} = 512 + 2.2 \times (-73) + \frac{1}{2} \times 2.2 \times 1.2 \times (-20) + \frac{1}{6} \times 2.2 \times 1.2 \times 0.2 \times 10$$

$$= 512 - 160.6 - 26.4 + 0.88$$

$$= 325.88$$

$$= 325 \text{ approx.}$$

3. To interpolate l_{47}:

Here $m = \frac{47-20}{10} = 2.7$

∴ From (1)

$$l_{47} = 512 + 2.7 \times (-73) + \frac{1}{2} \times 2.7 \times 1.7 \times (-20) + \frac{1}{6} \times 2.7 \times 1.7 \times .7 \times 10$$

$$= 512 - 197.1 - 45.9 + 5.355 = 274.355 = 274$$

Example 12: ***Find the number of men getting wages between ₹ 10 and ₹ 15 from the following table:***

Wages in ₹	0-10	10-20	20-30	30-40
Frequency	9	30	35	42

Solution: The difference table is as under:

Wages in ₹ x	Frequency $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
Under 10	9			
		30		
Under 20	39		5	
		35		2
Under 30	74		7	
		42		
Under 40	116			

Here, $a = 10, h = 10, \therefore\ m = \dfrac{x-a}{h} = \dfrac{15-10}{10} = 0.5$

∴ By Newton-Gregory interpolation formula,

$$f(15) = f(10) + (.5)\,\Delta f(10) + \frac{(0.5)(-0.5)}{\underline{|2}}\Delta^2 f(10) + \frac{(0.5)(-0.5)(-1.5)}{\underline{|3}}\Delta^3 f(10)$$

$$f(15) = 9 + 0.5\times 30 + \frac{(0.5)(-0.5)}{\underline{|2}}\times 5 + \frac{(0.5)(-0.5)(-1.5)}{\underline{|3}}\times 2$$

$$= 9 + 15 - 0.625 + 0.125$$

$$= 23.5$$

$$= 24 \text{ approx.}$$

∴ number of persons getting wages between ₹ 10 and ₹ 15.

$$= f(15) - f(10)$$

$$= 24 - 9 = 15.$$

Example 13: ***From the following table, find the number of students who obtained less then 45 marks.***

Marks :	30-40	40-50	50-60	60-70	70-80
No. of students :	31	42	51	35	31

Solution: The table of finite difference is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Below 40	31				
		42			
Below 50	73		9		
		51		– 25	
Below 60	124		– 16		37
		35		12	
Below 70	159		– 4		
		31			
Below 80	190				

Newton-Gregory formula for forward interpolation is

$$f(a+mh) = f(a) + {}^{m}C_1 \Delta f(a) + {}^{m}C_2 \Delta^2 f(a) + \ldots \qquad \ldots(1)$$

Here $h = 10, a = 40$

We have to interpolate $f(45)$

Thus, $a + mh = 45 \quad \Rightarrow \quad 40 + m.10 = 45$

$\Rightarrow$ $m = 0.5 = 1/2$

$$\therefore \quad f(45) = f(40) + \frac{1}{2}\Delta f(40) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{\underline{|2}}\Delta^2 f(40)$$

$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|3}}\Delta^3 f(40) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{\underline{|4}}\Delta^4 f(40)$$

$$= 31 + \frac{1}{2}(42) - \frac{1}{8}(9) - \frac{1}{16}(-25) - \frac{5}{128}(3)$$

$$= 47.868$$

Problem Set

1. Obtain the missing terms in the following table:

x :	0	.1	.2	.3	.4	.5	.6
$f(x)$:	.135	?	.111	.100	?	.082	.074

[B.C.A. (Avadh) 2009]

2. Find the first term of the series whose second and subsequent term are :

8, 3, 0, 1, – 1, 0

3. (i) Given $U_0 = 3, U_1 = 12, U_2 = 81, U_3 = 200, U_4 = 100, U_5 = 8$; find $\Delta^5 U_0$.

(ii) Given $U_0 = 1, U_1 = 11, U_2 = 21, U_3 = 28$ and $U_4 = 29$; find $\Delta^4 U_0$.

4. Given $U_0 = 580, U_1 = 556, U_2 = 520$ and $U_4 = 385$; find U_3.

5. Given $U_0 + U_8 = 1.9243, U_1 + U_7 = 1.9590$

$U_2 + U_6 = 1.9823, U_3 + U_5 = 1.9956$

Find U_4.

[B.C.A. (Meerut) 2003]

6. By the method of interpolation, estimate the missing value in the following table of rice cultivation giving area in 10,00,000 hectares:

Year	1911	1912	1913	1914	1915
Area	76.6	78.7	?	77.7	78.7
Year	1916	1917	1918	1919	
Area	79.5	80.6	77.6	78.7	

7. Find by constructing the difference table:

(i) Eighth term of the series

1, 1.095, 1.179, 1.251, 1.310, ...

(ii) Tenth term of the series

3, 14, 39, 84, 155, 258, ...

8. Prove that Newton-Gregory interpolation formula can be put in form:

$$f(x) = f(0) + x\Delta f(0) - xa\Delta^2 f(0) + xab\Delta^3 f(0) - xabc\Delta^4 f(0) + \ldots$$

Where $a = 1 - \dfrac{x+1}{2}, b = 1 - \dfrac{x+1}{3}, c = 1 - \dfrac{x+1}{4}$ etc.

9. Write down the polynomial of lowest degree which satisfies the following set of number:

0, 7, 26, 63, 124, 215, 314, 511.

10. Find the polynomial which fits the data in the following table:

x :	3	5	7	9	11
$y = f(x)$:	6	24	58	108	174

11. From the following table of yearly premiums for policies maturing at quinquennial ages. Interpolate the premium for policies maturing at the age of 12 years.

Age (years)	10	15	20	25	30	35
Premium $f(x)$	3.54	3.22	2.91	2.60	2.31	2.04

12. Using Newton-Gregory formula for interpolation to find the net premium at age 25 from the table given below:

Age	20	24	28	32
Net Premium	0.01427	0.01581	0.01772	0.01996

13. The population of a country in the decennial census were as under. Estimate the population for the year 1905.

Year x	1891	1901	1911	1921	1931
Population $f(x)$	98,752	132,285	168,076	195,690	246,050

Answers

1.	$f(0.1) = 0.123,\ f(0.4) = 0.090$
2.	15
3.	(i) 755 (ii) 0
4.	465
5.	0.9999
6.	76.16
7.	(i) 1.399 (ii) 1110
9.	$x^3 + 3x^2 + 3x$, which is the required polynomial.
10.	$2x^2 - 7x + 9$
11.	$f(12) = 3.41$ (Rounded up value)
12.	0.0162544 approx.
13.	147841

○○○

Unit-2

Chapter 4

Interpolation with Unequal Intervals

4.1 Divided Differences

Let $f(x_0), f(x_1), \ldots., f(x_n)$ be the entries corresponding to the arguments $x_0, x_1, \ldots., x_n$ where the intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ may not be equal, *i.e.*, not necessarily equally spaced. Then the first divided difference of $f(x)$ for the arguments x_0, x_1 is defined as

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{or} \quad \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

and is denoted by $f(x_0, x_1)$ or by $\underset{x_1}{\Delta} f(x_0)$

Similarly, the other first divided differences of $f(x)$ for the arguments $x_1, x_2, x_3, \ldots., x_{n-1}, x_n$ are

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \underset{x_2}{\Delta} f(x_1)$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \underset{x_3}{\Delta} f(x_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \underset{x_n}{\Delta} f(x_{n-1})$$

The second divided difference of $f(x)$ for the three arguments x_0, x_1 and x_2 is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \underset{x_1, x_2}{\Delta^2} f(x_0)$$

The *n*th divided difference is given by

$$f(x_0, x_1, x_2, \ldots., x_n) = \frac{f(x_1, x_2, \ldots., x_n) - f(x_0, x_1, \ldots., x_{n-1})}{x_n - x_0}$$

$$= \frac{f(x_0, x_1, \ldots., x_{n-1}) - f(x_1, x_2, \ldots.., x_n)}{x_0 - x_n} = \underset{x_1, x_2, \ldots., x_n}{\Delta^{n-1}} f(x_0)$$

Note: If two of the arguments coincide, the divided difference can be given a meaning assigned by taking the limit. Thus,

$$f(x_0, x_0) = \lim_{\in \to 0} f(x_0, x_0 + \in) = \lim_{\in \to 0} \frac{f(x_0 + \in) - f(x_0)}{\in} = f'(x_0),$$

if $f(x)$ is differentiable.

Similarly, $$f(x_0, x_0, x_0) = \frac{1}{2!} f''(x_0), \ldots, \underbrace{f(x_0, x_0, \ldots, x_0)}_{(r+1) \text{ arguments}} = \frac{1}{r!} f^{(r)}(x_0).$$

4.1.1 Divided Difference Table

A useful table called the divided difference table showing the values of the argument x, the entries and the divided differences is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$	$f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$	$f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$	
x_1	$f(x_1)$	$f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$	$f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$	$f(x_0, x_1, x_2, x_3) = \dfrac{\left\{\begin{matrix} f(x_1, x_2, x_3) \\ - f(x_0, x_1, x_2) \end{matrix}\right\}}{x_3 - x_0}$
x_2	$f(x_2)$	$f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$	$f(x_2, x_3, x_4) = \dfrac{f(x_3, x_4) - f(x_2, x_3)}{x_4 - x_2}$	$f(x_1, x_2, x_3, x_4) = \dfrac{\left\{\begin{matrix} f(x_2, x_3, x_4) \\ - f(x_1, x_2, x_3) \end{matrix}\right\}}{x_4 - x_1}$
x_3	$f(x_3)$	$f(x_3, x_4) = \dfrac{f(x_4) - f(x_3)}{x_4 - x_3}$		

4.1.2 Properties of Divided Differences

Theorems

Theorem 1: The divided differences are symmetric functions of their arguments that is, the value of any divided difference is independent of the order of the arguments. **[B.C.A. (Lucknow) 2006, 2010]**

Proof: We have, $$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} = \Sigma \frac{f(x_0)}{x_0 - x_1}$$

showing that $f(x_0, x_1)$ is symmetrical in x_0, x_1.

Again, $$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)}\left[\left\{\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}\right\} - \left\{\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}\right\}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)[(x_1 - x_0) - (x_1 - x_2)]}{(x_2 - x_0)(x_1 - x_2)(x_1 - x_0)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

or $$f(x_0, x_1, x_2) = \Sigma \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

showing that $f(x_0, x_1, x_2)$ is symmetrical in x_0, x_1, x_2.

Let us assume similar symmetrical expressions for the $(n-1)$th divided differences *i.e.*, let us assume that

$$f(x_0, x_1, \ldots, x_{n-1}) = \frac{f(x_0)}{(x_0 - x_1)\ldots.(x_0 - x_{n-1})} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots.(x_1 - x_{n-1})} + \ldots.$$

$$+ \frac{f(x_{n-1})}{(x_{n-1} - x_0)(x_{n-1} - x_1)\ldots.(x_{n-1} - x_{n-2})}$$

$$= \Sigma \frac{f(x_0)}{(x_0 - x_1)\ldots.(x_0 - x_{n-1})}$$

and similar expressions for the other $(n-1)$th divided differences.

Then, $$f(x_0, x_1, \ldots., x_n) = \frac{f(x_0, \ldots., x_{n-1}) - f(x_1, \ldots., x_n)}{x_0 - x_n}$$

$$= \frac{1}{(x_0 - x_n)}\left[\left\{\frac{f(x_0)}{(x_0 - x_1)\dots(x_0 - x_{n-1})} + \frac{f(x_1)}{(x_1 - x_0)\dots(x_1 - x_{n-1})} + \dots \right.\right.$$

$$\left. + \frac{f(x_{n-1})}{(x_{n-1} - x_0)\dots\dots(x_{n-1} - x_{n-2})}\right\}$$

$$\left.- \left\{\frac{f(x_1)}{(x_1 - x_2)\dots\dots(x_1 - x_n)} + \frac{f(x_2)}{(x_2 - x_1)\dots(x_2 - x_n)} + \dots\dots + \frac{f(x_n)}{(x_n - x_1)\dots\dots(x_n - x_{n-1})}\right\}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)\dots\dots(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)\dots\dots(x_1 - x_n)} + \dots\dots + \frac{f(x_n)}{(x_n - x_0)\dots\dots(x_n - x_{n-1})}$$

$$= \Sigma \frac{f(x_0)}{(x_0 - x_1)\dots\dots(x_0 - x_n)}$$

Showing that the *n*th divided difference $f(x_0, x_1, \dots., x_n)$ is also symmetrical in $x_0, x_1, \dots., x_n$ and thus completing the proof of the theorem by mathematical induction.

Thus, we see that a divided difference is a symmetrical function of all the arguments involved and it follows that for any function $f(x)$ the value of a divided difference depends only on the values of the arguments involved and not on the order in which they are taken. Thus,

$$f(x_0, x_1) = f(x_1, x_0),\ f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_2, x_1, x_0) = \dots. \text{ and so on.}$$

Theorem 2: The *n*th divided differences of a polynomial of the *n*th degree are constant. **[B.C.A. (Kanpur) 2005, 2008]**

Proof: First consider the function $f(x) = x^n$. The first divided differences of this function are given by

$$f(x_r, x_{r+1}) = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r} = \frac{x_{r+1}^n - x_r^n}{x_{r+1} - x_r}$$

$$= x_{r+1}^{n-1} + x_r x_{r+1}^{n-2} + \dots. + x_r^{n-2} x_{r+1} + x_r^{n-1},$$

which is a homogeneous expression of degree $(n - 1)$ in x_r, x_{r+1}.

[Note that $a^n - b^n = (a - b)\ (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots. + ab^{n-2} + b^{n-1})$]

The second divided differences are given by

$$f(x_r, x_{r+1}, x_{r+2}) = \frac{f(x_r, x_{r+1}) - f(x_{r+1}, x_{r+2})}{x_r - x_{r+2}}$$

$$= \frac{f(x_{r+1}, x_{r+2}) - f(x_r, x_{r+1})}{x_{r+2} - x_r}$$

$$= \frac{1}{(x_{r+2} - x_r)}[(x_{r+2}^{n-1} + x_{r+1}x_{r+2}^{n-2} + \dots. + x_{r+1}^{n-2}x_{r+2} + x_{r+1}^{n-2})$$

$$- (x_{r+1}^{n-1} + x_r x_{r+1}^{n-2} + \dots. + x_r^{n-2}x_{r+1} + x_r^{n-1})]$$

$$= \frac{x_{r+2}^{n-1} - x_r^{n-1}}{x_{r+2} - x_r} + x_{r+1}\frac{x_{r+2}^{n-2} - x_r^{n-2}}{x_{r+2} - x_r} + \ldots + x_{r+1}^{n-2}\frac{(x_{r+2} - x_r)}{x_{r+2} - x_r}$$

$$= (x_{r+2}^{n-2} + \ldots + x_r^{n-2}) + x_{r+1}(x_{r+1}^{n-3} + \ldots + x_r^{n-3}) + \ldots + x_{r+1}^{n-2},$$

which is a homogeneous expression of degree $n-2$ in x_r, x_{r+1} and x_{r+2}.

By induction it can be shown that $f(x_r, x_{r+1}, \ldots, x_{r+m})$ is a homogeneous expression of degree $n-m$. In particular, the *n*th divided difference of $f(x) = x^n$ is an expression of degree zero, *i.e.*, is a constant, and is therefore independent of the values of x_r, x_{r+1}, $x_{r+2}, \ldots, x_{r+n}$.

Since the *n*th divided differences of x^n are constant, therefore, the divided differences of x^n of order greater than *n* will all be zero.

If $f(x) = ax^n$, where *a* is a constant, then the *n*th divided difference of $f(x)$.

$= a$ (the *n*th divided difference of x^n), which is a constant.

Hence, if $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial of degree *n*, then the *n*th divided differences of all the terms except a_0x^n will be zero and so the *n*th divided difference of the whole polynomial will be constant.

Theorem 3: The *n*th divided difference can be expressed as the quotient of two determinants each of order *n* + 1.

Proof: Let us consider the third divided difference.

We have, $$f(x_0, x_1, x_2, x_3) = \Sigma \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{\Sigma\,[f(x_0)\,(\text{difference product of } x_1, x_2, x_3)]}{\text{difference product of } x_0, x_1, x_2, x_3}$$

By the theorem of determinants due to Vander-Monde, the difference product of

$$x_1, x_2, x_3 = \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}$$

Therefore, $$f(x_0, x_1, x_2, x_3) = \Sigma\left[f(x_0)\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}\right] + \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

or $$f(x_0, x_1, x_2, x_3) = \begin{vmatrix} f(x_0) & f(x_1) & f(x_2) & f(x_3) \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Thus, the third divided difference has been expressed as the quotient of two determinants each of order 4.

Proceeding the same way, we can express the higher order differences as the quotient of two determinants each of order one more than the order of the difference.

Theorem 4: The divided differences can be expressed as the product of multiple integral *i.e.*,

$$f(x_1, x_2, \ldots, x_n) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{n-2}} f^{n-1}(u_n)\, dt_{n-1}$$

Where $u_n = (1 - t_1)\, x_1 + (t_1 - t_2)\, x_2 + \ldots + (t_{n-2} - t_{n-1})\, x_{n-1} + t_{n-1} x_n, t_1, t_2, t_3, \ldots, t_n$ **are independent variables and** f^{n-1} **means the** $(n - 1)$**th derivative of** f.

Proof: First, we shall prove the result for $n = 2$ and $n = 3$.

For $n = 2$, the R.H.S. $= \int_0^1 f'(u_2)\, dt_1 = \int_0^1 f'\{(1 - t_1)\, x_1 + t_1 x_2\}\, dt_1$

$$= \left[\frac{f\{(1 - t_1)\, x_1 + t_1 x_2\}}{x_2 - x_1}\right]_0^1$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2) = \text{the L.H.S.}$$

Again for $n = 3$, the R.H.S.

$$= \int_0^1 dt_1 \int_0^{t_1} f''(u_3)\, dt_2$$

$$= \int_0^1 dt_1 \int_0^{t_1} f''\{(1 - t_1)\, x_1 + (t_1 - t_2)\, x_2 + t_2 x_3\}\, dt_2$$

$$= \int_0^1 dt_1 \left[f'\{(1 - t_1)\, x_1 + (t_1 - t_2)\, x_2 + t_2 x_3\}\right]_0^{t_1} \Big/ (x_3 - x_2)$$

$$= \frac{1}{x_3 - x_2} \int_0^1 f'\{(1 - t_1)\, x_1 + t_1 x_3\}\, dt_1 - \int_0^1 f'\{(1 - t_1)\, x_1 + t_1 x_2\}\, dt_1$$

$$= \frac{1}{(x_3 - x_2)(x_3 - x_1)}[f(x_3) - f(x_1)] - \frac{1}{(x_3 - x_2)(x_2 - x_1)}[f(x_2) - f(x_1)]$$

$$= \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)}$$

$$= f(x_1, x_2, x_3) = \text{the L.H.S.}$$

Thus, the result is true for $n = 2$ and $n = 3$.

Now let the result be true for n arguments. We shall show that it is also true for $(n + 1)$ arguments.

We have, $\int_0^{t_{n-1}} f^n(u_{n+1})\, dt_n$

$$= \int_0^{t_{n-1}} f^n\{(1 - t_1)\, x_1 + (t_1 - t_2)\, x_2 + \ldots + (t_{n-1} - t_n)\, x_n + t_n x_{n+1}\}\, dt_n$$

$$= \frac{1}{x_{n+1} - x_n} \left[f^{n-1}\{(1-t_1)\,x_1 + (t_1 - t_2)\,x_2 + \dots + (t_{n-1} - t_n)x_n + t_n x_{n+1}\} \right]_0^{t_{n-1}}$$

$$= \frac{1}{x_n - x_{n+1}} [f^{n-1}\{(1-t_1)x_1 + \dots. + (t_{n-2} - t_{n-1})x_{n-1} + t_{n-1}x_n\}$$

$$- f^{n-1}\{(1-t_1)x_1 + \dots. + (t_{n-2} - t_{n-1})x_{n-1} + t_{n-1}x_{n+1}\}]$$

$$\therefore \quad \int_0^1 dt_1 \int_0^{t_1} dt_2 \; \dots.. \int_0^{t_{n-1}} f^n(u_{n+1})\, dt_n$$

$$= \frac{f(x_1, x_2, \dots., x_n) - f(x_1, x_2, \dots, x_{n-1}, x_{n+1})}{x_n - x_{n+1}}$$

$$= \frac{f(x_n, x_2, \dots., x_1) - f(x_n, x_2, \dots, x_{n-1}, x_{n+1})}{x_1 - x_{n+1}} \qquad \text{[interchanging } x_1 \text{ and } x_n\text{]}$$

$$= \frac{f(x_1, x_2, \dots, x_n) - f(x_2, x_3, \dots, x_n, x_{n+1})}{x_1 - x_{n+1}} = f(x_1, x_2, \dots, x_{n+1})$$

Thus, the result is true for $(n + 1)$ arguments. We have already shown that the result is true for 2 and 3 arguments. Hence, by mathematical induction the result is true for n arguments where n is any +ive integer.

4.2 Newton's Formula for Unequal Intervals

Let $f(x_0), f(x_1), \dots., f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0, x_1, \dots., x_n$ not necessarily equally spaced. From the definition of divided differences,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

or $$f(x) = f(x_0) + (x - x_0)\, f(x, x_0) \qquad \dots(1)$$

Also $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

or $$f(x, x_0) = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1) \qquad \dots(2)$$

Similarly $$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2)\, f(x, x_0, x_1, x_2) \qquad \dots(3)$$

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$$f(x, x_0, x_1, \dots, x_{n-1}) = f(x_0, x_1, \dots, x_n) + (x - x_n)\, f(x, x_0, x_1, \dots, x_n) \qquad \dots(4)$$

Multiplying the equation (2) by $(x - x_0)$, (3) by $(x - x_0)\,(x - x_1)$ and so on and finally the equation (4) by $(x - x_0)\,(x - x_1)\, \dots.\, (x - x_{n-1})$ and adding to the equation (1), we have

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)\,(x - x_1)\, f(x_0, x_1, x_2) + \dots$$

$$+ (x - x_0)\,(x - x_1) \dots.\, (x - x_{n-1})\, f(x_0, x_1, \dots., x_n) + R_n,$$

where the remainder R_n is given by

$$R_n = (x - x_0)\,(x - x_1)\, \dots.\, (x - x_n)\, f(x, x_0, x_1, \dots., x_n).$$

Assuming that $f(x)$ is a polynomial of degree n, $f(x, x_0, x_1, \ldots, x_n)$ vanishes so that

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + \ldots$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, f(x_0, x_1, \ldots, x_n) \quad \ldots(5)$$

This formula is called **Newton's divided difference interpolation** formula.

4.2.1 Relation Between Divided Differences and Ordinary Differences

Let the arguments $x_0, x_1, x_2, \ldots, x_n$ be equally spaced,

i.e., $h = x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1}$

and let $x = x_0 + uh$. Then,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \Delta f(x_0),$$

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} \cdot \frac{\frac{1}{h} \Delta f(x_0) - \frac{1}{h} \Delta f(x_1)}{-2h}$$

$$= \frac{1}{2!\, h^2} \Delta^2 f(x_0).$$

Similarly, $f(x_0, x_1, \ldots, x_n) = \dfrac{1}{n!\, h^n} \Delta^n f(x_0)$

Substituting these values of the divided differences in Newton's formula *i.e.*, the equation (5), we get

$$f(x_0 + uh) = f(x_0) + \frac{uh}{1!\, h} \Delta f(x_0) + \frac{uh\,(uh - h)}{2!\, h^2} \Delta^2 f(x_0) + \ldots$$
$$+ \frac{uh\,(uh - h)(uh - 2h) \ldots (uh - \overline{n-1}\, h)}{n!\, h^n} \Delta^n f(x_0)$$

$$= f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0) + \ldots$$
$$+ \frac{u(u-1)(u-2) \ldots (u - \overline{n-1})}{n!} \Delta^n f(x_0)$$

$$= f(x_0) + {}^uC_1 \Delta f(x_0) + {}^uC_2 \Delta^2 f(x_0) + \ldots + {}^uC_n \Delta^n f(x_0),$$

which is Newton's formula for advancing differences.

4.3 Lagrange's Interpolation Formula for Unequal Intervals

[B.B.A. (Meerut) 2011; B.C.A. (Bhopal) 2008, 2012]

Let $f(x)$ denote a polynomial of the nth degree which takes the values $y_0, y_1, y_2, \ldots, y_n$ when x has the values $x_0, x_1, x_2, \ldots, x_n$ respectively. Then the $(n+1)$th divided differences of this polynomial are zero. Hence,

$$f(x, x_0, x_1, x_2, \ldots, x_n) = 0 \quad \ldots(1)$$

But we have, $f(x, x_0, x_1, x_2, \ldots, x_n)$

$$= \frac{y}{(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)} + \frac{y_0}{(x_0 - x)(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

$$+ \frac{y_1}{(x_1 - x)(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots + \frac{y_n}{(x_n - x)(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

$\therefore$ using (1), we get $\dfrac{y}{(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)}$

$$+ \frac{y_0}{(x_0 - x)(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} + \frac{y_1}{(x_1 - x)(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots$$

$$+ \frac{y_n}{(x_n - x)(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} = 0$$

Transposing all the terms except the first to the right hand side, we get

$$\frac{y}{(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)}$$

$$= \frac{y_0}{(x - x_0)(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} + \frac{y_1}{(x - x_1)(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots$$

$$+ \frac{y_n}{(x - x_n)(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}$$

Solving for y and then removing the common factors $(x - x_0), (x - x_1), (x - x_2), \ldots,$ $(x - x_n)$ in the respective terms, we get

$$y = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_n)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\ldots(x_2 - x_n)} y_2 + \ldots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\ldots(x_n - x_{n-1})} y_n \qquad \ldots(2)$$

This is Lagrange's formula and is seen to give $y = y_0, y_1, \ldots, y_n$ when $x = x_0, x_1, \ldots, x_n$ respectively. The values of the independent variable may or may not be equidistant.

Since Lagrange's formula is merely a relation between two variables, either of which may be taken as the independent variable, it is evident that by considering y as the independent variable we can write a formula giving x as a function of y. Hence, on interchanging x and y in (2), we get

$$x = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)} x_1$$

$$+ \frac{(y - y_0)(y - y_1)\ldots(y - y_n)}{(y_2 - y_0)(y_2 - y_1)\ldots(y_2 - y_n)} x_2 + \ldots$$

$$+ \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots(y_n - y_{n-1})} x_m \qquad \ldots(3)$$

The main uses of Lagrange's formula are: (i) to find any value of a function when the given values of the independent variable are not equidistant and (ii) to find the value of the independent variable corresponding to a given value of the function.

The reader will notice that Lagrange's formula is tedious to apply and involves a great deal of computation. It must also be used with care and caution, for if the values of the independent variable are not taken close together the results are liable to be very inaccurate.

SOLVED EXAMPLES

Example 1: ***Construct a divided difference table for the following:***

x :	1	2	4	7	12
$f(x)$:	22	30	82	106	216

[B.B.A. (Meerut) 2008]

Solution: The divided difference table for the given data is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{(-3.6)-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{0.535-(-1.6)}{12-1}$ $=0.194$
		$\frac{106-82}{7-4}=8$		$\frac{1.75-(-3.6)}{12-2}$ $=0.535$	
7	106		$\frac{22-8}{12-4}=1.75$		
		$\frac{216-106}{12-7}=22$			
12	216				

Example 2: ***Find the third divided difference with arguments*** $2, 4, 9, 10$ ***of the function*** $f(x) = x^3 - 2x$.

[B.C.A. (Kurukshetra) 2012]

Solution: The divided difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\frac{56-4}{4-2}=26$		
4	56		$\frac{131-26}{9-2}=15$	
		$\frac{711-56}{9-4}=131$		$\frac{23-15}{10-2}=1$
9	711		$\frac{269-131}{10-4}=23$	
		$\frac{980-711}{10-9}=269$		
10	980			

Hence, the third divided difference is 1.

Example 3: ***Find out the divided differences of y_x, given that:***

x :	1	2	4	7	12
y_x :	22	30	82	106	216

Solution: The divided difference table is as given below:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-3.6-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{.535-(-1.6)}{12-1}$ $=.1940$
		$\frac{106-82}{7-4}=8$		$\frac{1.75-(-3.6)}{12-2}$ $=.535$	
7	106		$\frac{22-8}{12-4}=1.75$		
		$\frac{216-106}{12-7}=22$			
12	216				

Example 4: *If* $y_{35.0} = 1175,\ y_{35.5} = 1280,\ y_{39.65} = 2180,\ y_{40.5} = 2420$, *find* y_{40} *by divided differences.*

Solution: The divided difference table is as given below:

x	y_x	Δy_x	$\Delta^2\ y_x$	$\Delta^3\ y_x$
35	1175			
		$\frac{1280-1175}{35.5-35} = 210$		
35.5	1280		$\frac{225-210}{39.5-35} = 3.33$	
		$\frac{2180-1280}{39.5-35.5} = 225$		$\frac{3-3.33}{40.5-35} = -.06$
39.5	2180		$\frac{240-225}{40.5-35.5} = 3$	
		$\frac{2420-2180}{40.5-39.5} = 240$		
40.5	2420			

By Newton's divided difference formula for arguments x_0, x_1, x_2, x_3, we get

$$y_x = y(x_0) + (x - x_0)\ y(x_0, x_1) + (x - x_0)(x - x_1)\ y(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\ y(x_0, x_1, x_2, x_3).$$

Substituting the values from the table and taking $x = 40$, we get

$$y_{40} = 1175 + (40 - 35) \times 210 + (40 - 35)(40 - 35.5) \times 3.33$$

$$+ (40 - 35)(40 - 35.5)(40 - 39.5) \times (-0.06)$$

$$= 1175 + 1050 + 74.925 - 0.675$$

$$= 2299.25$$

Example 5: *By means of Newton's divided difference formula find the values of* $f(8)$ *and* $f(15)$ *from the following table:*

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Solution: The divided difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Using Newton's interpolation formula for unequal intervals *i.e.*, Newton's divided difference formula, we get

$$f(8) = 48 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)\times 1$$

$$= 448,$$

and $$f(15) = 48 + (15-4)\times 52 + (15-4)(15-5)\times 15 + (15-4)(15-5)(15-7)\times 1$$

$$= 3150$$

Example 6: ***Given that $f(0) = 8$, $f(1) = 68$, $f(5) = 123$, construct a divided difference table, using the table determine the value of $f(2)$.***

Solution: The divided difference table is as given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	8		
		$\frac{68-8}{1-0}=60$	
1	68		$\frac{13.75-60}{5-0}=-9.25$
		$\frac{123-68}{5-1}=13.75$	
5	123		

By Newton's divided difference formula, we get

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$

$$\therefore \quad f(2) = 8 + (2 - 0) \times 60 + (2 - 0)(2 - 1)(-9.25)$$

$$= 109.50$$

Example 7: ***If $f(x) = \frac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$ and $f(a, b, c, d)$.***

[B.C.A. (Kashi) 2008]

Solution: We have, $f(a, b) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a}$

$$= -\frac{(b^2 - a^2)}{(b - a)\, b^2 a^2} = -\frac{b + a}{a^2 b^2} \qquad \ldots(1)$$

Again $\quad f(a, b, c) = \dfrac{f(b, c) - f(a, b)}{c - a}$

$$= \frac{1}{c - a}\left[-\frac{b + c}{b^2 c^2} + \frac{a + b}{a^2 b^2}\right], \qquad \text{using (1)}$$

$$= -\frac{1}{(c - a)}\left[\frac{b + c}{b^2 c^2} - \frac{a + b}{a^2 b^2}\right]$$

$$= -\frac{1}{(c - a)}\left[\frac{a^2(b + c) - c^2(a + b)}{a^2 b^2 c^2}\right]$$

$$= -\left[\frac{b(a^2 - c^2) + ac\,(a - c)}{(c - a)\, a^2 b^2 c^2}\right]$$

$$= \frac{ab + bc + ca}{a^2 b^2 c^2} \qquad \ldots(2)$$

Now, $\quad f(a, b, c, d) = \dfrac{f(b, c, d) - f(a, b, c)}{d - a}$

$$= \frac{1}{(d - a)}\left[\frac{bc + cd + db}{b^2 c^2 d^2} - \frac{ab + bc + ca}{a^2 b^2 c^2}\right] \qquad \text{using (2)}$$

$$= \frac{1}{(d - a)}\left[\frac{a^2(bc + cd + db) - d^2(ab + bc + ca)}{a^2 b^2 c^2 d^2}\right]$$

$$= \frac{1}{(d - a)}\left[\frac{bc\,(a^2 - d^2) + acd\,(a - d) + abd\,(a - d)}{a^2 b^2 c^2 d^2}\right]$$

$$= \frac{(a - d)}{(d - a)}\left[\frac{bc\,(a + d) + acd + abd}{a^2 b^2 c^2 d^2}\right]$$

$$= -\frac{abc + bcd + acd + abd}{a^2 b^2 c^2 d^2}.$$

Example 8: **Show that** $\underset{bcd}{\Delta}^3 \left(\frac{1}{a}\right) = -\frac{1}{abcd}$.

Solution: The divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a}$			
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$		
b	$\frac{1}{b}$		$(-1)\frac{\frac{1}{bc}-\frac{1}{ab}}{c-a} = (-1)^2\frac{1}{abc}$	
		$\frac{\frac{1}{c}-\frac{1}{b}}{c-b} = -\frac{1}{bc}$		$(-1)^2\frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = (-1)^3\frac{1}{abcd}$
c	$\frac{1}{c}$		$(-1)\frac{\frac{1}{dc}-\frac{1}{bc}}{d-b} = (-1)^2\frac{1}{bcd}$	
		$\frac{\frac{1}{d}-\frac{1}{c}}{d-c} = -\frac{1}{dc}$		
d	$\frac{1}{d}$			

Hence, $$\underset{bcd}{\Delta}^3 \frac{1}{a} = (-1)^3 \frac{1}{abcd} = -\frac{1}{abcd}.$$

Example 9: **Show that** $\underset{yz}{\Delta}^2 x^3 = x + y + z$.

Solution: We construct the following divided difference table:

Argument	Entry	First Divide Differences	Second Divided Differences
x	x^3		
		$\frac{y^3 - x^3}{y - x} = y^2 + x^2 + xy$	
y	y^3		$\frac{(z^2 + y^2 + zy) - (y^2 + x^2 + yx)}{z - x}$ $= \frac{(z^2 - x^2) + y(z - x)}{z - x}$
		$\frac{z^3 - y^3}{z - y} = z^2 + y^2 + zy$	
z	z^3		$= x + y + z$

From the table, we observe that

$$\underset{yz}{\Delta}^2 x^3 = x + y + z.$$

Example 10: ***Find the polynomial of the lowest possible degree which assumes the values* 3, 12, 15, – 21 *when x has the values* 3, 2, 1, – 1 *respectively.***

Solution: For the given data the divided difference table is as given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
– 1	– 21			
		$\frac{15+21}{1+1}=18$		
1	15		$\frac{-3-18}{2+1}=-7$	
		$\frac{12-15}{2-1}=-3$		$\frac{-3+7}{3+1}=1$
2	12		$\frac{-9+3}{3-1}=-3$	
		$\frac{3-12}{3-2}=-9$		
3	3			

By Newton's divided difference formula, we get

$$f(x) = f(x_0) + (x - x_0)\ \Delta f(x_0) + (x - x_0)(x - x_1)\ \Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\ \Delta^3 f(x_0)$$

or $$f(x) = -21 + \{x - (-1)\}(18) + \{x - (-1)\}(x - 1)(-7) + \{x - (-1)\}(x - 1)(x - 2)(1)$$
$$= -21 + (x + 1)18 + (x + 1)(x - 1)(-7) + (x + 1)(x - 1)(x - 2)$$
$$= x^3 - 9x^2 + 17x + 6.$$

Example 11: ***Find the form of the function*** $y = U_x$ ***given that***

$$U_0 = 8, U_1 = 11, U_4 = 68, U_5 = 123.$$ **[B.C.A. (Indore) 2012]**

Solution: The divided difference table is as given below:

x	U_x	ΔU_x	$\Delta^2 U_x$	$\Delta^3 U_x$
0	8			
		$\frac{11-8}{1-0}=3$		
1	11		$\frac{19-3}{4-0}=4$	
		$\frac{68-11}{4-1}=19$		$\frac{9-4}{5-0}=1$
4	68		$\frac{55-19}{5-1}=9$	
		$\frac{123-68}{5-4}=55$		
5	123			

By Newton's divided difference formula, we have

$$\begin{aligned} U_x &= 8 + (x-0)\cdot 3 + (x-0)(x-1)\cdot 4 + (x-0)(x-1)(x-4)\cdot 1 \\ &= 8 + 3x + 4x^2 - 4x + x^3 - 5x^2 + 4x \\ &= x^3 - x^2 + 3x + 8. \end{aligned}$$

Example 12: ***Given the values:***

x :	4	5	7	10	11	13
u_x :	48	100	294	900	1210	2028

From the table of divided differences and extend is to include the values x = 2 and x = 15.

Solution: The divided difference table is as follows:

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$
2	4					
		22				
4	48		10			
		52		1		
5	100		15		0	
		97		1		0
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		0
11	1210		33		0	
		409		1		
13	2028		38			
		561				
15	3150					

$$\therefore \quad \begin{aligned} u_x &= 48 + 52(x-4) + 15(x-4)(x-5) + 1(x-4)(x-5)(x-7) \\ &= 48 + 52(x-4) + 15(x^2 - 9x + 20) + (x^3 - 16x^2 + 83x - 140) \\ &= x^3 - x^2 = x^2(x-1) \end{aligned}$$

$$\therefore \quad u_2 = 2^3 - 2^2 = 4 \qquad \text{and} \quad u_{15} = 15^3 - 15^2 = 3150.$$

The original divided difference table is for the arguments values 4 to 13, leaving 2 and 15 and the figures in antigue.

The complete table, with arguments 2 and 15 and the figures in antigue, gives the original divided difference table extended to include the values $x = 2$ and $x = 15$.

Example 13: ***Find a polynomial satisfied by** $(-4, 1245), (-1, 33), (0, 5), (2, 9)$ **and** $(5, 1335)$ **by the use of Newton's interpolation formula with divided differences.***

OR

***Find the polynomial of the lowest possible degree which assumes the values** $1245, 33, 5, 9$ **and** 1335 **at** $x = -4, -1, 0, 2$ **and** 5 **respectively. Also find the value of the polynomial at the abscissa** 1.*

Solution: For the given values the divided difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{14+13}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

Using the values from above table in Newton's divided difference formula, we have

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)\cdot(94) + (x+4)(x+1)(x).(-14) + (x+4)(x+1)(x)(x-2)\,3$$

$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

For abscissa $x = 1$, we have

$$f(1) = 3 - 5 + 6 - 14 + 5 = -5.$$

Example 14: ***Find the polynomial of fifth degree from the following data:***

$$u_0 = -18,\ u_1 = 0,\ u_3 = 0,\ u_5 = -248,\ u_6 = 0,\ u_9 = 13104.$$

[B.C.A. (Lucknow) 2007]

Solution: From the given data we see that the value of the function is zero at $x = 1, 3, 6$. Let the function $f(x)$ be a polynomial in x. Then $(x-1), (x-3)$ and $(x-6)$ will be factors of $f(x)$.

Let $\quad f(x) = (x-1)(x-3)(x-6)\,p(x)$. ...(1)

Since we are given 6 values, so we can fit a polynomial of fifth degree. As such $p(x)$ will be a polynomial of second degree in x.

From (1), $$p(x) = \frac{f(x)}{(x-1)(x-3)(x-6)}$$

$\therefore$ $$p(0) = \frac{f(0)}{(0-1)(0-3)(0-6)} = \frac{-18}{-18} = 1.$$

$$p(5) = \frac{f(5)}{(5-1)(5-3)(5-6)} = \frac{-248}{-8} = 31.$$

and $$p(9) = \frac{f(9)}{(9-1)(9-3)(9-6)} = \frac{13104}{144} = 91.$$

Now we shall find $p(x)$. The divided difference table for $p(x)$ is as follows:

x	p	$\Delta p(x)$	$\Delta^2 p(x)$
0	1		
		6	
5	31		1
		15	
9	91		

By Newton's divided difference formula, we get

$$p(x) = p(0) + (x-0)\,\Delta p(0) + (x-0)(x-5)\,\Delta^2 p(0)$$

$$= 1 + x.6 + x(x-5)\cdot = x^2 + x + 1.$$

Thus, $$f(x) = (x-1)(x-3)(x-6)(x^2 + x + 1)$$

$$= x^5 - 9x^4 + 18x^3 - x^2 + 9x - 18.$$

Example 15: ***If the data are $u_0, u_3, u_4, u_7, u_{11}$ and the interpolation formula is***

$$u_x = u_4 + C_1 \underset{3}{\Delta} u_4 + C_2 \underset{4,7}{\Delta^2} u_3 + C_3 \underset{3,4,7}{\Delta^3} u_0 + C_4 \underset{3,4,7,11}{\Delta^4} u_0$$

find the values of C_1, C_2, C_3 and C_4.

Solution: Newton's divided difference formula is

$$u_x = u(x_0) + (x - x_0)\,u(x_0, x_1) + (x - x_0)(x - x_1)\,u(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,u(x_0, x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,u(x_0, x_1, x_2, x_3, x_4)$$

$\therefore$ $$u_x = u_4 + (x-4)\underset{3}{\Delta} u_4 + (x-4)(x-3)\underset{4,7}{\Delta^2} u_3$$
$$+ (x-7)(x-4)(x-3)\underset{3,4,7}{\Delta^3} u_0 + (x-0)(x-7)(x-4)(x-3)\underset{3,4,7,11}{\Delta^4} u_0.$$

comparing it with the given result, we get

$$C_1 = x - 4,\ C_2 = (x-4)(x-3),\ C_3 = (x-7)(x-4)(x-3),$$
$$C_4 = x(x-7)(x-4)(x-3).$$

Example 16: *Using the following table, find $f(x)$ as a polynomial in powers of $(x-6)$.*

x :	-1	0	2	3	7	10
$f(x)$:	-11	1	1	1	141	561

Solution: The divided difference table is as given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-11			
		$\frac{1+11}{0+1}=12$		
0	1		$\frac{0-12}{2+1}=-4$	
		$\frac{1-1}{2-0}=0$		$\frac{0+4}{3+1}=1$
2	1		0	
		$\frac{1-1}{3-2}=0$		$\frac{7-0}{7-0}=1$
3	1		$\frac{35-0}{7-2}=7$	
		$\frac{141-1}{7-3}=35$		$\frac{15-7}{10-2}=1$
7	141		$\frac{140-35}{10-3}=15$	
		$\frac{561-141}{10-7}=140$		$\frac{a-15}{6-3}=1$
10	561		$\frac{b-140}{6-7}=a$	
....	....	....	....	....
		$\frac{c-561}{6-10}=b$		$\frac{a'-a}{6-7}=1$
6	c		$\frac{b'-b}{6-10}=a'$	
		b'		$\frac{a''-a'}{6-10}=1$
6	c		a''	
		b''		
6	c			

Since the third differences are constant, $f(x)$ is a polynomial of degree three and hence the argument 6 should appear 3 times successively in the difference table. The extended table has been shown below the dotted line.

from the column of third differences,

$$\frac{a-15}{6-3}=1 \qquad \text{or} \quad a=15+3=18.$$

Also
$$\frac{a'-a}{6-7}=1 \qquad \Rightarrow a'=a-1=17,$$

and
$$\frac{a''-a'}{6-10}=1 \qquad \Rightarrow a''=a'-4=13.$$

From the column of second differences, we have

$$\frac{b-140}{6-7}=a \qquad \Rightarrow \quad b=140-a=122,$$

and
$$\frac{b'-b}{6-10}=a' \qquad \Rightarrow \quad b'=b-4a'=54.$$

Again from the column of first differences, we have

$$\frac{c-561}{6-10}=b \qquad \Rightarrow \quad c=561-4b=73.$$

Newton's formula with arguments x_0, x_1, x_2, x_3 is

$$f(x)=f(x_0)+(x-x_0)\,f(x_0,x_1)+(x-x_0)(x-x_1)\,f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)\,f(x_0,x_1,x_2,x_3)$$

Putting $x_0=x_1=x_2=x_3=6$, we get

$$f(x)=f(6)+(x-6)\,b'+(x-6)^2\,a''+(x-6)^3\times 1$$
$$=73+54\,(x-6)+13\,(x-6)^2+(x-6)^3.$$

Example 17: ***Apply Lagrange's formula to find the cubic polynomial which includes the following values of x and*** y_x:

x :	0	1	4	6
y_x :	1	– 1	1	– 1

Solution: By Lagrange's interpolation formula for the given values, we have

$$y_x=\frac{(x-1)(x-4)(x-6)}{(0-1)(0-4)(0-6)}\cdot(1)+\frac{(x-0)(x-4)(x-6)}{(1-0)(1-4)(1-6)}\cdot(-1)$$
$$+\frac{(x-0)(x-1)(x-6)}{(4-0)(4-1)(4-6)}\cdot(1)+\frac{(x-0)(x-1)(x-4)}{(6-0)(6-1)(6-4)}\cdot(-1)$$
$$=\frac{x^3-11x^2+34x-24}{-24}-\frac{x^3-10x^2+24x}{15}+\frac{x^3-7x^2+6x}{-24}$$
$$-\frac{x^3-5x^2+4x}{60}$$
$$=-\frac{17x^3}{120}+\frac{29x^2}{24}+\frac{207x}{120}+\frac{5}{4}.$$

Example 18: ***In the following table h is the height above the sea level and p the barometric pressure; calculate p when h = 5280.***

h :	0	4763	6942	10594
p :	27	25	23	20

Solution: Here, $x_0=0, x_1=4763, x_2=6942, x_3=10594$ and $f(x_0)=27,\ f(x_1)=25,$ $f(x_2)=23, f(x_3)=20.$

Applying Lagrange's interpolation formula taking $x = 5280$, we have

$$f(5280) = \frac{(5280-4763)(5280-6942)(5280-10594)}{(0-4763)(0-6942)(0-10594)} \times 27$$

$$+ \frac{(5280-0)(5280-6942)(5280-10594)}{(4763-0)(4763-6942)(4763-10594)} \times 25$$

$$+ \frac{(5280-0)(5280-4763)(5280-10594)}{(6942-0)(6942-4763)(6942-10594)} \times 23$$

$$+ \frac{(5280-0)(5280-4763)(5280-6942)}{(10594-0)(10594-4763)(10594-6942)} \times 20$$

or $$f(5280) = 0.34437 + 19.26325 + 6.03776 - 0.402204 = 24.80.$$

Example 19: **Given, $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$,**
$\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$
Find **$\log_{10} 656$.**

Solution: For the values x_0, x_1, x_2, x_3 Lagrange's formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3).$$

Here, $x_0 = 654, x_1 = 658, x_2 = 659, x_3 = 661, x = 656$. Substituting these values in Lagrange's formula, we get

$$\log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202.$$

$$\therefore \quad \log_{10} 656 = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{(2)(-3)(-5)}{(4)(-1)(-3)} \times 2.8182$$

$$+ \frac{(2)(-2)(-5)}{(5)(1)(-2)} \times 2.8189 + \frac{(2)(-2)(-3)}{(7)(3)(2)} \times 2.8202$$

$$= \frac{3 \times 2.8156}{14} + \frac{5 \times 2.8182}{2} - \frac{2 \times 2.8189}{1} + \frac{2 \times 2.8202}{7}$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$

$$= 2.8168.$$

Example 20: ***The values of y and x are given as below:***

x :	5	6	9	11
y :	12	13	14	16

Find the value of y when $x = 10$.

Solution: Applying Lagrange's formula for $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$ and $x = 10$, we get

$$y_{10} = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)} \times 12 + \frac{(5)(1)(-1)}{(1)(-3)(-5)} \times 13 + \frac{(5)(4)(-1)}{(4)(3)(-2)} \times 14 + \frac{(5)(4)(1)}{(6)(5)(2)} \times 16$$

$$= 2 - 4.33 + 11.66 + 5.33 = 14.66.$$

Example 21: ***Apply Lagrange's formula to find $f(1.50)$, using the following values of the function $f(x)$:***

x :	1.00	1.20	1.40	1.60	1.80	2.00
$f(x)$:	0.2420	0.1942	0.1497	0.1109	0.0790	0.0540

[B.C.A. (Kanpur) 2002, 2006, 2011]

Solution: Using the Lagrang's interpolation formula for the given values, we have

$$f(1.50) = \frac{(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-1.8)(1.5-2.0)}{(1-1.2)(1-1.4)(1-1.6)(1-1.8)(1-2)} \times 0.2420$$

$$+ \frac{(1.5-1)(1.5-1.4)(1.5-1.6)(1.5-1.8)(1.5-2.0)}{(1.2-1)(1.2-1.4)(1.2-1.6)(1.2-1.8)(1.2-2)} \times 0.1942$$

$$+ \frac{(1.5-1)(1.5-1.2)(1.5-1.6)(1.5-1.8)(1.5-2)}{(1.4-1)(1.4-1.2)(1.4-1.6)(1.4-1.8)(1.4-2)} \times 0.1497$$

$$+ \frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.8)(1.5-2)}{(1.6-1)(1.6-1.2)(1.6-1.4)(1.6-1.8)(1.6-2)} \times 0.1109$$

$$+ \frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-2)}{(1.8-1)(1.8-1.2)(1.8-1.4)(1.8-1.6)(1.8-2)} \times 0.0790$$

$$+ \frac{(1.5-1)(1.5-1.2)(1.5-1.4)(1.5-1.6)(1.5-1.8)}{(2-1)(2-1.2)(2-1.4)(2-1.6)(2-1.8)} \times 0.0540$$

$$= \frac{0.00001089}{0.0384} - \frac{0.00014565}{0.00768} + \frac{0.000336825}{0.00384} + \frac{0.000249525}{0.00384} - \frac{0.00005925}{0.00768}$$

$$+ \frac{0.0000243}{0.0384}$$

$$= 0.000283593 - 0.018964843 + 0.087714843 + 0.064980468 - 0.007714843 + 0.000632812$$

$$= 0.12693$$

Example 22: ***The following table gives the normal weights of babies during the first twelve months of life:***

Age in months (x):	0	2	5	8	10	12
Height in lbs $f(x)$:	$7\frac{1}{2}$	$10\frac{1}{4}$	15	16	18	21

Estimate the weight of the baby at the age of 7 months.

Solution: Here, $x_0 = 0, x_1 = 2, x_2 = 5, x_3 = 8, x_4 = 10, x_5 = 12$. By Lagrange's formula for interpolation, we have

$$f(7) = \frac{(7-2)(7-5)(7-8)(7-10)(7-12)}{(0-2)(0-5)(0-8)(0-10)(0-12)} \times \frac{15}{2}$$
$$+ \frac{(7-0)(7-5)(7-8)(7-10)(7-12)}{(2-0)(2-5)(2-8)(2-10)(2-12)} \times \frac{41}{4}$$
$$+ \frac{(7-0)(7-2)(7-8)(7-10)(7-12)}{(5-0)(5-2)(5-8)(5-10)(5-12)} \times 15$$
$$+ \frac{(7-0)(7-2)(7-5)(7-10)(7-12)}{(8-0)(8-2)(8-5)(8-10)(8-12)} \times 16$$
$$+ \frac{(7-0)(7-2)(7-5)(7-8)(7-12)}{(10-0)(10-2)(10-5)(10-8)(10-12)} \times 18$$
$$+ \frac{(7-0)(7-2)(7-5)(7-8)(7-10)}{(12-0)(12-2)(12-5)(12-8)(12-10)} \times 21$$

$$= \frac{15}{128} - \frac{287}{384} + 5 + \frac{175}{12} - \frac{63}{16} + \frac{21}{32}$$

$$= 0.11718 - 0.7473 + 5 + 14.583 - 3.937 + 0.652$$

$$= 15.67 \text{ lbs.}$$

Example 23: ***Apply Lagrange's formula to find $f(5)$ given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$ and explain why the result differs from 2^5?***

Solution: Given, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$ and $f(x_0) = 2$, $f(x_1) = 4$, $f(x_2) = 8$, $f(x_3) = 16$, $f(x_4) = 128$.

∴ from Lagrange's formula for interpolation, we have

$$f(5) = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)} \times 2 + \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)} \times 4$$
$$+ \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)} \times 8 + \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)} \times 16$$
$$+ \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)} \times 128$$

$$= -\frac{2}{3} + \frac{32}{5} - 24 + \frac{128}{3} + \frac{128}{15}$$

$$= -0.66 + 6.4 - 2.4 + 42.66 + 8.53$$

$$= 57.59 - 24.66 = 32.93.$$

From the given values of function we observe that the function is of the form 2^x *i.e.*, $f(x) = 2^x$ giving $f(5) = 2^5 = 32$ which differs from the interpolated value 32.93 due to the fact that for applying Lagrange's formula we have assumed the given function to be expressible as a polynomial while actually the given function is of the form 2^x.

Example 24: ***For the following table find the form of the function $f(x)$:***

x :	0	1	2	5
$f(x)$:	2	3	12	147

Solution: Here, $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

by Lagrange's formula, we have

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

Substituting the values of x_0, x_1, x_2, x_3 in this, we get

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

or $$\frac{f(x)}{x(x-1)(x-2)(x-5)} = -\frac{1}{5x} + \frac{3}{4(x-1)} - \frac{2}{x-2} + \frac{49}{20(x-5)}$$

$$= \frac{20x^3 + 20x^2 - 20x + 40}{20x(x-1)(x-2)(x-5)}$$

$\Rightarrow$ $$f(x) = x^3 + x^2 - x + 2.$$

Example 25: ***The following table is given***

x :	0	1	2	3	4
$f(x)$:	3	6	11	18	27

What is the form of the function $f(x)$?

Solution: Proceeding as in the above example, we get $\dfrac{f(x)}{x(x-1)(x-2)(x-3)(x-4)}$

$$= \frac{3}{(-1)(-2)(-3)(-4)} \cdot \frac{1}{x} + \frac{6}{1(-1)(-2)(-3)} \cdot \frac{1}{(x-1)} + \frac{11}{2.1.(-1)(-2)} \cdot \frac{1}{(x-2)}$$

$$+ \frac{18}{3.2.1.(-1)} \cdot \frac{1}{(x-3)} + \frac{27}{4.3.2.1} \cdot \frac{1}{(x-4)}$$

$$= \frac{1}{8x} - \frac{1}{(x-1)} + \frac{11}{4(x-2)} - \frac{3}{(x-3)} + \frac{9}{8(x-4)}$$

$$= \frac{x^2 + 2x + 3}{x(x-1)(x-2)(x-3)(x-4)}$$

$$\therefore \qquad f(x) = x^2 + 2x + 3$$

Problem Set

1. From the following table find $f(x)$ in powers of $(x-3)$:

x :	5	11	27	34	42
$f(x)$:	23	899	17315	35606	68510

[B.C.A. (Rohilkhand) 2003]

2. Show that Lagrange's interpolation formula can be evolved by equating $(n+1)$th divided differences of $f(x)$ to zero if $f(x)$ is a polynomial of degree n.

3. The mode of a certain frequency curve $y = f(x)$ is very near $x = 9$ and the values of the frequency density $f(x)$ for $x = 8.9, 9.0$ and 9.3 are respectively equal to 0.30, 0.35 and 0.25 calculate the approximate value of the mode.

4. The following values of the function $f(x)$ for values of x are given $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$ find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

5. By means of Lagrange's formula, prove that approximately

$$y_0 = \frac{1}{2}(y_1 - y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

6. By means of Lagrange's formula, prove that $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2$ $(y_{-3} - y_{-5})$ approximately.

7. Four equidistant values u_{-1}, u_0, u_1 and u_2 being given a value is interpolated by Lagrange's formula. Show that it may be written in the form

$$u_x = yu_0 + xu_1 + y\frac{(y^2-1)}{3!}\Delta^2_{u-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0.$$

where $\qquad x + y = 1.$

8. If all terms except y_5, of the sequence $y_1, y_2, \ldots, y_9$ be given, show that the value of y_5 is

$$\frac{56\,(y_4 + y_6) - 28\,(y_3 + y_7) + 8\,(y_2 + y_8) - (y_1 + y_9)}{70}$$

9. The values of $f(x)$ are given at a, b and c. Show that the maximum or minimum is attained by

$$x = \frac{f(a)\,(b^2 - c^2) + f(b)\,(c^2 - a^2) + f(c)\,(a^2 - b^2)}{2\,\{f(a)\,(b - c) + f(b)\,(c - a) + f(c)\,(a - b)\}}$$

10. Prove that the Lagrange's formula can be put in the form

$$p_n(x) = \sum_{r=0}^{n} \frac{\phi(x)\,f(x_r)}{(x - x_r)\,\phi'(x_r)}$$

where, $\phi(x) = \prod_{r=0}^{n} (x - x_r)$.

[B.C.A. (Kurukshetra) 2011]

11. Values of y_x are given for all integral values of x from 0 to $n - 1$. Show that y_x is capable of being expressed in the form:

$$\frac{x\,!}{(x-n)\,!\,(n-1)\,!}\left[\frac{y_{n-1}}{x-n+1} - {}^{n-1}C_1\,\frac{y_{n-2}}{x-n+2} + {}^{n-1}C_2\,\frac{y_{n-3}}{x-n+3} + \ldots \pm {}^{n-1}C_{n-1}\,\frac{y_0}{x}\right]$$

12. Given the following data, find $f(x)$ as a polynomial in powers of $(x - 5)$, by extending the table to include argument $x = 5$ repeated as many times as may be necessary:

$$f(0) = 4,\ f(2) = 26,\ f(3) = 58,\ f(4) = 112,\ f(7) = 466,\ f(9) = 922$$

Also find $f'(5)$, $f''(5)$ and $f'''(5)$.

13. The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable?

Answers

1.	$f(x) = (x-3)^3 + 6\,(x-3)^2 + 2\,(x-3) - 13$ which is the required polynomial in powers of $(x-3)$.
2.	$f(x) = \dfrac{(x-x_1)\ldots(x-x_n)}{(x_0-x_1)\ldots.(x_0-x_n)} f(x_0) + \ldots + \dfrac{(x-x_0)\ldots(x-x_{n-1})}{(x_n-x_0)\ldots(x_n-x_{n-1})} f(x_n)$ which is Lagrange's interpolation formula.
3.	Hence, the mode of the frequency curve $y = f(x)$ is 9.07.
4.	Since $f''(x) = \frac{1}{6} \times (-2)$ which is < 0, therefore, $f(x)$ is maximum at the point $x = 4.5$.
5.	$y_0 = \frac{1}{2}(y_1 - y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$
6.	$y_1 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5})$
7.	$u_x = yu_0 + xu_1 + \dfrac{y\,(y^2-1)}{3\,!}\Delta^2 u_{-1} + \dfrac{x\,(x^2-1)}{3\,!}\Delta^2 u_0$
8.	$y_3 = 0.05\,(y_0 + y_6) - 0.3\,(y_1 + y_5) + 0.75\,(y_2 + y_4)$
9.	$x = \dfrac{(b^2-c^2)f(a) + (c^2-a^2)\,f(b) + (a^2-b^2)\,f(c)}{2\,\{(b-c)\,f(a) + (c-a)\,f(b) + (a-b)\,f(c)\}}$
10.	$p_n(x) = \sum\limits_{r=0}^{n} \dfrac{\phi(x)\,f(x_r)}{(x-x_r)\,\phi'(x_r)}$
12.	$f'(5) = f(5,5) = b' = 98$ $f''(5) = 2\,!,\ f(5,5,5) = 2\,!, a'' = 2 \times 17 = 34$ and $f'''(5) = 3\,!\,f(5,5,5,5) = 3\,!.1 = 6$
13.	147

❍❍❍

Unit-2

Chapter 5

Central Difference Interpolation Formulae

5.1 Introduction

Newton's formulae which have been already discussed are fundamental and are applicable to nearly all cases of interpolation, but they do not converge as rapidly as the central difference formulae. These formulae are based on differences obtained from the values of $f(x)$ on either side of the origin.

The ***central difference operator*** (δ) is defined by the operator equation

$$\delta = E^{1/2} - E^{-1/2}.$$

[B.C.A. (Meerut) 2004, 2006; B.C.A. (Agra) 2007, 2011; B.C.A. (Kanpur) 2010; B.C.A. (Kashi) 2012]

$\therefore$ the first central difference $\delta\, f(x) = f\left(x + \dfrac{h}{2}\right) - f\left(x - \dfrac{h}{2}\right)$

$$= (E^{1/2} - E^{-1/2})\, f(x) \text{ where } E \equiv 1 + \Delta.$$

$$\therefore \qquad \delta = E^{1/2} - E^{-1/2} = E^{-1/2}\,(E - 1) = E^{-1/2}\,\Delta. \qquad \text{...(1)}$$

Again the first backward or ascending difference $\nabla f(x)$ of $f(x)$ is defined as

$$\nabla f(x) = f(x) - f(x - h) = (1 - E^{-1})\, f(x).$$

i.e. $$\nabla \equiv (1 - E^{-1}) \quad \text{or} \quad \nabla E^{1/2} \equiv E^{1/2} - E^{-1/2} = \delta. \qquad \text{...(2)}$$

Thus from (1) and (2), we have $\delta = \Delta\, E^{-1/2} = \nabla\, E^{1/2}$. ...(3)

The nth power of these operators applied to y_x gives

$$\delta^n\, y_x = \Delta^n\, E^{-n/2}\, y_x = \nabla^n\, E^{n/2}\, y_x$$

or $$\delta^n\, y_x = \Delta^n\, y_{x-n/2} = \nabla^n\, y_{x+n/2}.$$

The average operator μ is defined by $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$.

[B.C.A. (Lucknow) 2003, 2008]

$$\therefore \qquad \mu\, y_x = \frac{1}{2}[y_{x+1/2} + y_{x-1/2}] \text{ and } \mu\,\delta^n y_x = \frac{1}{2}[\delta^n y_{x+1/2} + \delta^n y_{x-1/2}].$$

$$\text{Again} \qquad \mu^2\, y_x = \frac{1}{4}[E^{1/2} + E^{-1/2}]^2\, y_x = \frac{1}{4}[(E^{1/2} - E^{-1/2})^2 + 4]\, y_x$$

$$= \frac{1}{4}(\delta^2 + 4)\, y_x = \left(1 + \frac{1}{4}\delta^2\right) y_x. \quad \therefore \quad \mu^2 \equiv 1 + \frac{1}{4}\delta^2. \qquad \text{...(4)}$$

$$\text{Again} \qquad \delta\, y_x = y_{x+1/2} - y_{x-1/2} \text{ and } \mu\, y_x = \frac{1}{2}(y_{x+1/2} + y_{x-1/2})$$

$$\therefore \qquad 2\mu\, y_x + \delta\, y_x = 2\, y_{x+1/2} \quad \text{or} \quad (2\mu + \delta)\, y_x = 2\, E^{1/2}\, y_x$$

$$\therefore \qquad 2\mu + \delta = 2E^{1/2} \quad \text{or} \quad E^{1/2} = \mu + \frac{1}{2}\delta. \qquad \text{...(5)}$$

The operator inverse to δ is called the *central sum operator* and is denoted by $\sigma\,(=\delta^{-1})$. It is defined by

$$\sigma\, f(x) = \sigma\, f(x-h) + f\left(x - \frac{1}{2}h\right) \text{ or } f(x) = \sigma\, f\left(x + \frac{1}{2}h\right) - \sigma\, f\left(x - \frac{1}{2}h\right)$$

$$\text{Also} \qquad (\sigma - \sigma E^{-1})\, f(x) = E^{-1/2}\, f(x) \Rightarrow \sigma = \frac{E^{1/2}}{E-1}.$$

[B.C.A. (Avadh) 2009]

Note: $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$ and $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

Similarly $\quad \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}, \Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc.

$\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}, \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

and also $\quad \Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

The appearance of a central difference table is as under:

x	y_x	$\delta\, y_x$	$\delta^2\, y_x$	$\delta^3\, y_x$	$\delta^4\, y_x$
-2	y_{-2}	$\Delta\, y_{-2} =$			
		$\delta\, y_{-3/2}$	$\Delta^2\, y_{-2} =$		
-1	y_{-1}	$\Delta\, y_{-1} =$	$\delta^2\, y_{-1}$	$\Delta^3\, y_{-2} =$	
		$\delta\, y_{-1/2}$	$\Delta^2\, y_{-1} =$	$\delta^3\, y_{-1/2}$	$\Delta^4\, y_{-2} =$
0	y_0	$\Delta\, y_0 =$	$\delta^2\, y_0$	$\Delta^3\, y_{-1} =$	$\delta^4\, y_0$
		$\delta\, y_{1/2}$	$\Delta^2\, y_0 =$	$\delta^3\, y_{1/2}$	
1	y_1	$\Delta\, y_1 =$	$\delta^2\, y_1$		
		$\delta\, y_{3/2}$			
2	y_2				

5.2 Gauss's Interpolation Formulae Or Gauss's Central Difference Formulae

[B.C.A. (Agra) 2008; B.C.A. (Kurukshetra) 2007, 2010, 2012]

1. **The general Newton formula gives**

$$
\begin{aligned}
y = f(x) = {} & f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) \\
& + (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) \\
& + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\, f(x_0, x_1, x_2, x_3, x_4) \\
& + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\, f(x_0, x_1, x_2, x_3, x_4, x_5) \\
& + (x - x_0)(x - x_1) \ldots (x - x_5)\, f(x_0, x_1, x_2, x_3, x_4, x_5, x_6) + \ldots
\end{aligned}
$$

Now putting $x_0 = x_0, x_1 = x_0 + h, x_2 = x_0 - h, x_3 = x_0 + 2h,$

$x_4 = x_0 - 2h, x_5 = x_0 + 3h, x_6 = x_0 - 3h$ etc., we immediately get

$$
\begin{aligned}
f(x) = {} & f(x_0) + (x - x_0)\, f(x_0, x_0 + h) \\
& + (x - x_0)(x - x_0 - h)\, f(x_0, x_0 + h, x_0 - h) \\
& + (x - x_0)(x - x_0 - h)(x - x_0 + h)\, f(x_0, x_0 + h, x_0 - h, x_0 + 2h) \\
& + (x - x_0)(x - x_0 - h)(x - x_0 + h)(x - x_0 + 2h) \\
& \qquad \times f(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h) \\
& + (x - x_0)(x - x_0 - h)(x - x_0 + h)(x - x_0 - 2h)(x - x_0 + 2h) \\
& \qquad \times f(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h, x_0 + 3h) + \ldots
\end{aligned}
$$

Further, let $u = (x - x_0)/h \Rightarrow x - x_0 = hu.$

$$
\begin{aligned}
\Rightarrow \quad f(x) = {} & f(x_0) + hu\, f(x_0, x_0 + h) + hu\,(hu - h)\, f(x_0 - h, x_0, x_0 + h) \\
& + hu\,(hu - h)(hu + h)\, f(x_0 - h, x_0, x_0 + h, x_0 + 2h) \\
& + hu\,(hu - h)(hu + h)(hu - 2h)\, f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) \\
& + hu\,(hu - h)(hu + h)(hu - 2h)(hu + 2h) \\
& \qquad \times f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h) + \ldots \qquad \ldots(1)
\end{aligned}
$$

But, we also know that

$$f(x_0, x_0 + h) = \frac{\Delta y_0}{h}, \quad f(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2 y_{-1}}{2!\,h^2},$$

$$f(x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^3 y_{-1}}{3!\,h^3},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^4 y_{-2}}{4!\,h^4},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h) = \frac{\Delta^5 y_{-2}}{5!\,h^5} \text{ etc.}$$

Now, substituting these values in (1), we readily obtain

$$y = y_0 + hu\,\frac{\Delta y_0}{h} + h^2 u(u-1)\,\frac{\Delta^2 y_{-1}}{2\,!h^2} + h^3 u(u-1)(u+1)\,\frac{\Delta^3 y_{-1}}{3\,!h^3}$$

$$+ h^4\, u(u-1)(u+1)(u-2)\,\frac{\Delta^4 y_{-2}}{4\,!h^4}$$

$$+ h^5\, u(u-1)(u+1)(u-2)(u+2)\,\Delta^5 y_{-2}/\{5\,!h^5\} + \ldots$$

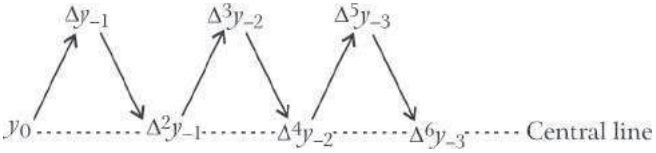
$$\Rightarrow \qquad y = y_0 + u\,\Delta y_0 + u(u-1)\,\frac{\Delta^2 y_{-1}}{2\,!} + u(u^2-1)\,\frac{\Delta^3 y_{-1}}{3\,!}$$

$$+ u(u^2-1)(u-2)\,\frac{\Delta^4 y_{-2}}{4\,!} + u(u^2-1)(u^2-2^2)\,\frac{\Delta^5 y_{-2}}{5\,!} + \ldots$$

$$\Rightarrow \qquad y = y_0 + u_{C_1}\,\Delta y_0 + u_{C_2}\,\Delta^2 y_{-1} + (u+1)_{C_3}\,\Delta^3 y_{-1} + (u+1)_{C_4}\,\Delta^4 y_{-2} + \ldots \qquad \ldots(2)$$

This result is known as ***Gauss's forward formula for equal intervals***. In the central difference notation, the formula reduces to

$$y = y_0 + u_{C_1}\,\delta y_{1/2} + u_{C_2}\,\delta^2 y_0 + (u+1)_{C_3}\,\delta^3 y_{1/2} + (u+1)_{C_4}\,\delta^4 y_0 + \ldots$$



The formula employs odd differences just below the central line and even differences on the central line as shown in the figure. It is used to interpolate the values of y for $u: 0 < u < 1$ measured forwardly from the origin.

2. **Gauss's Backward Formula:** Substituting $x_0 = x_0$, $x_1 = x_0 - h$, $x_2 = x_0 + h$, $x_3 = x_0 - 2h$, $x_4 = x_0 + 2h$, $x_5 = x_0 - 3h$, $x_6 = x_0 + 3h$ etc. in general Newton formula, we get

$$y = y_0 + (x-x_0)\,f(x_0, x_0-h) + (x-x_0)(x-x_0+h)\,f(x_0, x_0-h, x_0+h)$$

$$+ (x-x_0)(x-x_0+h)(x-x_0-h)\,f(x_0, x_0-h, x_0+h, x_0-2h)$$

$$+ (x-x_0)(x-x_0+h)(x-x_0-h)(x-x_0-2h) \times$$

$$f(x_0, x_0-h, x_0+h, x_0-2h, x_0+2h)$$

$$+ (x-x_0)(x-x_0+h)(x-x_0-h)(x-x_0+2h)(x-x_0-2h)$$

$$\times f(x_0, x_0-h, x_0+h, x_0-2h, x_0+2h, x_0-3h) + \ldots$$

Further, putting $u = \dfrac{x-x_0}{h}$ *i.e.,* $x - x_0 = uh$; we get

$$y_u = y_0 + hu\,f(x_0-h, x_0) + hu(hu+h)\,f(x_0-h, x_0, x_0+h)$$

$$+ hu(hu+h)(hu-h)\,f(x_0-2h, x_0-h, x_0, x_0+h)$$

$$+ hu(hu+h)(hu-h)(hu+2h)\,f(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h)$$

$$+ hu(hu+h)(hu-h)(hu+2h)(hu-2h)$$

$$\times f(x_0-3h, x_0-2h, x_0-h, x_0, x_0+h, x_0+2h) + \ldots$$

Again we know that

$$f(x_0 - h, x_0) = \frac{\Delta\, y_{-1}}{h}, \; f(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2\, y_{-1}}{2\,!\,h^2}$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h) = \frac{\Delta^3\, y_{-2}}{3\,!\,h^3},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^4\, y_{-2}}{4\,!\,h^4},$$

$$f(x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^5\, y_{-3}}{5\,!\,h^5} \text{ etc.}$$

$$\Rightarrow \qquad y_u = y_0 + u\,\Delta y_{-1} + u\,(u + 1)\,\frac{\Delta^2\, y_{-1}}{2\,!} + u\,(u^2 - 1)\,\frac{\Delta^3\, y_{-2}}{3\,!}$$

$$+ u\,(u^2 - 1)\,(u + 2)\,\frac{\Delta^4\, y_{-2}}{4\,!} + u\,(u^2 - 1)\,(u^2 - 2^2)\,\frac{\Delta^5\, y_{-3}}{5\,!} + \ldots$$

$$\Rightarrow \qquad y_u = y_0 + {}^{u}C_1\,\Delta\, y_{-1} + {}^{(u+1)}C_2\,\Delta^2\, y_{-1} + {}^{(u+1)}C_3\,\Delta^3\, y_{-2}$$

$$+ {}^{(u+2)}C_4\Delta^4\, y_{-2} + \ldots + {}^{(u+k-1)}C_{2k-1}\Delta^{2k-1}\, y_{-k} \qquad \ldots(1)$$

This result is known as ***Gauss's Backward Formula.***

Another formula called ***Gauss Backward Formula*** for equal intervals may be derived in a similar way; we have

$$\Delta y_0 = \Delta y_{-1} + \Delta^2\, y_{-1};\, \Delta^3\, y_{-1} = \Delta^3\, y_{-2} + \Delta^4\, y_{-2} \text{ and so on.}$$

Putting these values in (1) and grouping the terms in a slightly different way, we have

$$y_u = y_0 + {}^{u}C_1\Delta y_{-1} + {}^{u+1}\,C_2\Delta^2\, y_{-1} + {}^{u+1}C_3\Delta^3\, y_{-2}$$

$$+ {}^{u+2}C_4\Delta^4\, y_{-2} + \ldots + {}^{u+k-1}C_{2k-1}\,\Delta^{2k-1}\, y_{-k} + {}^{u+k}C_{2k}\,\Delta^{2k}\, y_{-k} + \ldots \qquad \ldots(2)$$

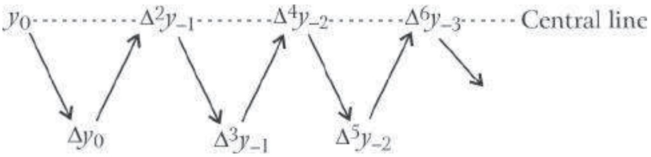
We observe that this formula *employs the odd differences just above the central line through* y_0 *and even differences on the central line.*

In the central differences notation, this formula reduces to

$$y_u = y_0 + u\delta\, y_{-1/2} + \frac{(u-1)\,u}{2\,!}\,\delta^2\, y_0 + \frac{(u+1)\,u\,(u-1)}{3\,!}\,\delta^3\, y_{-1/2}$$

$$+ \frac{(u+2)\,(u+1)\,u\,(u-1)}{4\,!}\,\delta^4\, y_0 + \ldots$$

The formula contains odd differences above the central line and even differences on the central line:



This formula is used to interpolate the values of y for a negative value of $u : -1 < u < 0$.

Gauss's forward and backward formulae are not of much practical importance. However, these serve as intermediate steps for obtaining some important formulae.

(i) Gauss Backward Formula.

In case, we take the differences just above and those lying on the horizontal line through $x = x_2$ *i.e.*

$f(x_2), \underset{x_2}{\Delta} f(x_1),\ \underset{x_2\,x_3}{\Delta^2} f(x_1),\ \underset{x_1\,x_2\,x_3}{\Delta^3} f(x_0)$ and $\underset{x_1\,x_2\,x_3\,x_4}{\Delta^4} f(x_0)$ and so on then by Shepard's rule, we have

$$f(x) = f(x_2) + (x - x_2) \underset{x_2}{\Delta} f(x_1), + (x - x_2)(x - x_1) \underset{x_2\,x_3}{\Delta^2} f(x_1)$$

$$+ (x - x_2)(x - x_1)(x - x_3) \underset{x_1\,x_2\,x_3}{\Delta^3} f(x_0) + (x - x_2)(x - x_1)(x - x_3)(x - x_0) \underset{x_1\,x_2\,x_3\,x_4}{\Delta^4} f(x_0) + \ldots$$

Putting $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$ and various divided differences in terms of ordinary differences, we get

$$f(x) = f(0) + x\Delta f(-1) + \frac{x(x+1)}{2!}\Delta^2 f(-1) + \frac{(x+1)\,x\,(x-1)}{3!}\Delta^3 f(-2)$$

$$+ \frac{(x+2)(x+1)\,x\,(x-1)}{4!}\Delta^4 f(-2) + \ldots$$

On further simplification, we get

$$f(x) = f(0) + {}^{x}C_1\Delta f(-1) + {}^{x+1}C_2\Delta^2 f(-1) + {}^{x+1}C_3\Delta^3 f(-2) + {}^{x+2}C_4\Delta^4 f(-2) + \ldots \qquad \ldots(1)$$

This is called ***Gauss backward formula.***

(ii) Third formula due to Gauss.

This formula starts with y_1 and runs parallel to the backward formula

We have $$u = \frac{x - x_0}{h} \Rightarrow \frac{x - x_k}{h} = \frac{x - (x_0 + kh)}{h} = u - k.$$

To obtain the formula, we increase the subscripts of x only in backward formula by one unit and change the *u's* by putting $k = 1$ in the formula $u - k = \frac{x - x_k}{h}$.

We thus get $x - x_1 = hu - h$ *i.e.* we have to put $u - 1$ for u.

Hence equation (2) reduce

$$\Rightarrow \qquad y_u = y_1 + (u-1)\,\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_{-2}}{3!}$$

$$+ u(u^2 - 1)(u-2)\frac{\Delta^4 y_{-1}}{4!} + u(u^2-1)(u-2)(u-3)\frac{\Delta^5 y_{-2}}{5!} + \ldots \qquad \ldots(1)$$

Aliter:

Taking $f(1)$ as the initial term and proceding as in Gauss backward formula, taking the successive differences in the following order:

$$f(x_3),\ \underset{x_3}{\Delta} f(x_2),\ \underset{x_3\,x_4}{\Delta^2} f(x_2),\ \underset{x_2\,x_3\,x_4}{\Delta^3} f(x_1),\ \underset{x_1\,x_2\,x_3\,x_4}{\Delta^4} f(x_0)$$

and so on, we get

$$f(x) = f(x_3) + (x - x_3) \underset{x_3}{\Delta} f(x_2) + (x - x_3)(x - x_2) \underset{x_3\,x_4}{\Delta^2} f(x_2)$$

$$+ (x - x_3)(x - x_2)(x - x_4) \underset{x_2\,x_3\,x_4}{\Delta^3} f(x_1)$$

$$+ (x - x_3)(x - x_2)(x - x_4)(x - x_1) \underset{x_1\,x_2\,x_3\,x_4}{\Delta^4} f(x_0) + \ldots \qquad \ldots(2)$$

Putting $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$ and replacing the various divided differences in terms of ordinary differences in (1), we get

$$f(x) = f(1) + (x-1)\,\Delta f(0) + \frac{(x-1)\,x}{2\,!}\,\Delta^2 f(0) + \frac{(x-1)\,x\,(x-2)}{3\,!}\,\Delta^3 f(-1)$$

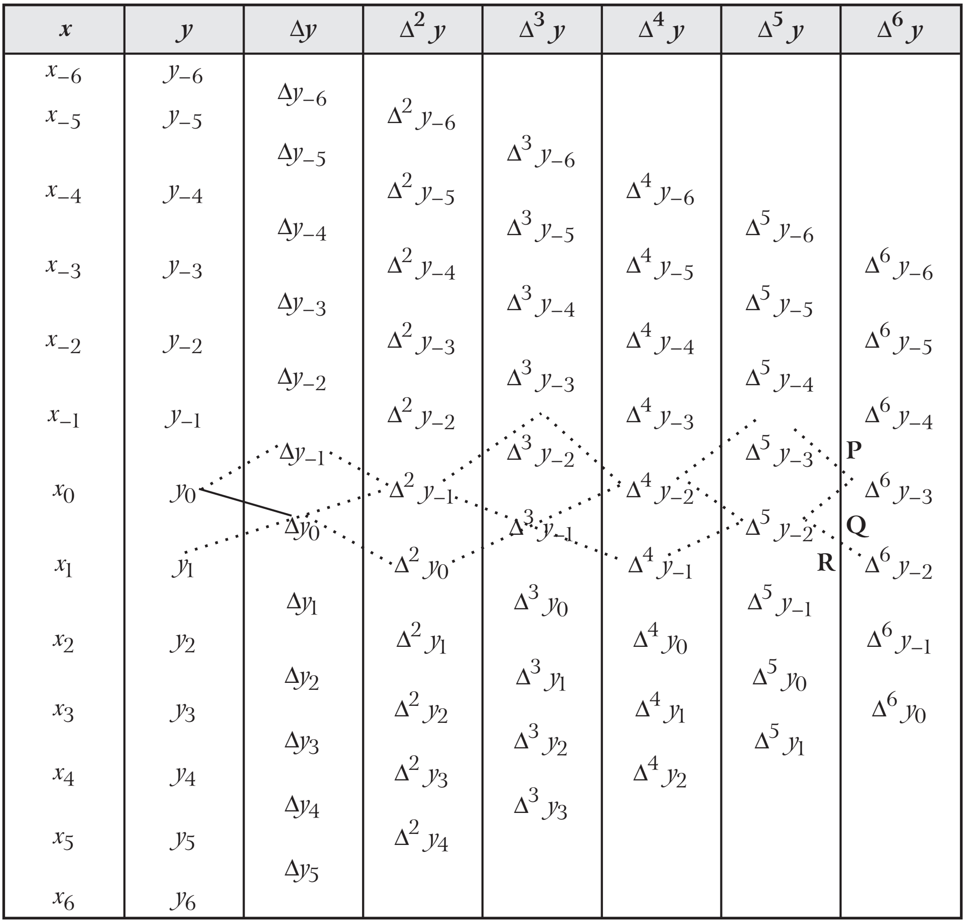
$$+ \frac{(x-2)(x+1)\,x\,(x-1)}{4\,!}\,\Delta^4 f(-2) + \ldots \qquad \ldots(3)$$

On further simplication, we get

$$f(x) = f(1) + {}^{x-1}C_1 \Delta f(0) + {}^{x}C_2 \Delta^2 f(0) + {}^{x}C_3 \Delta^3 f(-1) + {}^{x+1}C_4 \Delta^4 f(-2) + \ldots \qquad \ldots(4)$$

This is known as ***third formula due to Gauss.***

In the following difference table the paths of the three Gauss formulae have been shown. The designations P, Q, R correspond to [eqn. (2) eqn. (3) eqn. (4)]. Now refer table.



x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_{-6}	y_{-6}						
		Δy_{-6}					
x_{-5}	y_{-5}		$\Delta^2 y_{-6}$				
		Δy_{-5}		$\Delta^3 y_{-6}$			
x_{-4}	y_{-4}		$\Delta^2 y_{-5}$		$\Delta^4 y_{-6}$		
		Δy_{-4}		$\Delta^3 y_{-5}$		$\Delta^5 y_{-6}$	
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$		$\Delta^4 y_{-5}$		$\Delta^6 y_{-6}$
		Δy_{-3}		$\Delta^3 y_{-4}$		$\Delta^5 y_{-5}$	
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		$\Delta^6 y_{-5}$
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

5.3 Stirling's Formula

[B.C.A. (Rohtak) 2012; B.C.A. (Lucknow) 2006, 2008]

Taking the mean of the Gauss's formulae, we have

$$y_u = y_0 + u\,\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2\,!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3\,!}\cdot\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}$$

$$+ \frac{u^2(u^2-1^2)}{4\,!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5\,!}\cdot\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \ldots$$

where $u = (x - x_0)/h$.

This formula is known as stirling's formula. It employs the mean of the odd differences just above and just below the central line and even differences on the central line. The following difference table gives the position of the differences used in the formula:

	Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$	
y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$		$\Delta^8 y_{-4}$
	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$	

The most general formula is as follows:

$$y_u = y_0 + u\,\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{u^2}{2\,!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3\,!}\cdot\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{u^2(u^2-1^2)}{4\,!}\Delta^4 y_{-2} + \ldots$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)\ldots[u^2-(k-1)^2]}{(2k-1)\,!}\cdot\frac{\Delta^{2k-1} y_{-(k-1)} + \Delta^{2k-1} y_{-k}}{2}$$

$$+ \frac{u^2(u^2-1^2)(u^2-2^2)\ldots[u^2-(k-1)^2]}{(2k)\,!}\Delta^{2k} y_{-k} + \ldots \qquad \ldots(1)$$

In central differences notation, (1) takes the form

$$y_u = y_0 + u\mu\,\delta\, y_0 + \frac{u^2}{2\,!}\delta^2 y_0 + \frac{u(u^2-1^2)}{3\,!}\mu\,\delta^3 y_0 + \frac{u^2(u^2-1^2)}{4\,!}\delta^4 y_0 + \ldots \qquad \ldots(2)$$

$$\left[\because \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\,\delta\, y_0,\right.$$

$$\left.\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\,\delta^3\, y_0 \text{ etc.}\right]$$

This formula has means of the odd differences just below and above the central line and even differences on this line.

$$-\,y_0 \begin{pmatrix}\Delta y_{-1}\\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{pmatrix} \Delta^6 y_{-3} \quad \text{central line}$$

5.4 Bessel's Formula

Bessel's Interpolation Formula: Taking the means of Gauss's forward formula and the third formula due to Gauss's we obtain

$$y_u = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right) \Delta y_0 + \frac{u(u-1)}{2} \frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{u\left(u - \frac{1}{2}\right)(u-1)}{3\,!} \Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1)(u-2)}{4\,!} \frac{(\Delta^4 y_{-2} + \Delta^4 y_{-1})}{2} + \frac{u\left(u - \frac{1}{2}\right)(u^2-1)(u-2)}{5\,!} \Delta^5 y_{-2} + \ldots \qquad \ldots(1)$$

This is Bessel's formula and can be written in a slightly different form by transforming the first two terms to $y_0 + u\,\Delta y_0$ $[\because \quad \Delta y_0 = y_1 - y_0]$.

Then equation (1)

$$\Rightarrow \qquad y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2} \frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{u\left(u - \frac{1}{2}\right)(u-1)}{3\,!} \Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1)(u-2)}{4\,!} \frac{(\Delta^4 y_{-2} + \Delta^4 y_{-1})}{2} + \ldots \qquad \ldots(2)$$

Hence the more general form of Bessel's formula is given by

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2} \frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} + \frac{u\left(u - \frac{1}{2}\right)(u-1)}{3\,!} \Delta^3 y_{-1}$$

$$+ \frac{(u+1)\,u\,(u-1)(u-2)}{4\,!} \frac{(\Delta^4 y_{-1} + \Delta^4 y_{-2})}{2} + \frac{u\left(u - \frac{1}{2}\right)(u^2-1)(u-2)}{5\,!} \Delta^5 y_{-2}$$

$$+ \frac{u(u^2-1)(u^2-4)(u-3)}{6\,!} \frac{(\Delta^6 y_{-3} + \Delta^6 y_{-2})}{2} + \ldots$$

$$+ \frac{u(u^2-1)(u^2-4)\ldots(u-n)(u+n-1)}{(2n)\,!} \frac{(\Delta^{2n} y_{-n} + \Delta^{2n} y_{-n+1})}{2}$$

$$+ \frac{u\left(u - \frac{1}{2}\right)(u^2-1)(u^2-4)\ldots(u-n)(u+n-1)}{(2n+1)\,!} \Delta^{2n+1} y_{-n}. \qquad \ldots(3)$$

Now substituting $u = \frac{1}{2}$ in (3), we obtain **[Bessel's Formula]**

$$y_u = \frac{y_0 + y_1}{2} - \left(\frac{1}{8}\right) \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(\frac{3}{128}\right) \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} - \left(\frac{5}{1024}\right) \frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \ldots$$

$$+ (-1)^n \frac{[1.3.5\ldots(2n-1)]^2}{2^{2n}(2n)\,!} \frac{\Delta^{2n} y_{-n} + \Delta^{2n} y_{-n+1}}{2} \qquad \ldots(4)$$

This is the special case of Bessel's formula and is called the formula for interpolating to halves. It is used for computing the values of the function mid-way between any two given values.

Further, putting $u - \frac{1}{2} = v$ or $u = v + \frac{1}{2}$ in (3), we get a more convenient form of Bessel's formula [symmetrical, too]

$$y_u = \frac{y_0 + y_1}{2} + v\Delta y_0 + \frac{\left(v^2 - \frac{1}{4}\right)}{2\,!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{v\left(v^2 - \frac{1}{4}\right)}{3\,!} \Delta^3 y_{-1}$$

$$+ \frac{\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)}{4\,!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{v\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)}{5\,!} \Delta^5 y_{-2}$$

$$+ \ldots + \frac{\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)\ldots\left\{v^2 - \frac{(2n-1)^2}{4}\right\}}{(2n)\,!} \times \frac{\Delta^{2n} y_{-n} + \Delta^{2n} y_{-n+1}}{2}$$

$$+ \frac{v\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)\ldots\left\{v^2 - \frac{(2n-1)^2}{4}\right\}}{(2n+1)\,!} \Delta^{2n+1} y_{-n}.$$

The quantities that occur in Stirling's formula are shown in the following difference table:

y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$		$\Delta^8 y_{-4}$
	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$	
y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$

Now in the central differences notation, (3) takes the form

$$y_u = y_0 + u\delta\, y_{1/2} + \frac{u(u-1)}{2\,!}\mu\,\delta^2 y_{1/2} + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3\,!}\delta^3 y_{1/2}$$

$$+ \frac{(u+1)(u-1)(u-2)}{4\,!}\mu\,\delta^4 y_{1/2} + \ldots \qquad \ldots(5)$$

$$\left[\because\ \frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = \mu\,\delta^2 y_{1/2}, \frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\,\delta^4 y_{1/2} \text{ etc.}\right]$$

Above is a very useful formula for practical purposes. It has odd differences below the central line and means of even differences on and below this line:

$$\begin{array}{c} y_0\cdots\cdots\cdots \\ \\ \Delta y_0 \\ \\ \end{array} \left\{\begin{array}{c} \Delta^2 y_{-1} \\ \\ \\ \\ \Delta^2 y_0 \end{array}\right\} \begin{array}{c} \cdots\cdots\cdots \\ \\ \Delta^3 y_{-1} \\ \\ \end{array} \left\{\begin{array}{c} \Delta^4 y_{-2} \\ \\ \\ \\ \Delta^4 y_{-1} \end{array}\right\} \begin{array}{c} \cdots\cdots\cdots \\ \\ \Delta^5 y_{-2} \\ \\ \end{array} \left\{\begin{array}{c} \Delta^6 y_{-3} \\ \\ \\ \\ \Delta^6 y_{-2} \end{array}\right\} \begin{array}{l} \ldots \text{ central line} \\ \\ \\ \\ \end{array}$$

5.5 Laplace-Everett Formula

The Gauss forward formula is

$$y_u = y_0 + u\Delta\, y_0 + u\,(u-1)\,\frac{\Delta^2 y_{-1}}{2\,!} + u\,(u^2-1)\,\frac{\Delta^3 y_{-1}}{3\,!}$$

$$+\, u\,(u^2-1)\,(u-2)\,\frac{\Delta^4 y_{-2}}{4\,!} + u\,(u^2-1)\,(u^2-2^2)\,\frac{\Delta^5 y_{-2}}{5\,!} + \ldots \qquad \ldots(1)$$

But, we have $\Delta^{2k+1} y_{-k} = \Delta^{2k}(\Delta y_{-k}) = \Delta^{2k}(y_{-k+1} - y_{-k}) = \Delta^{2k} y_{1-k} - \Delta^{2k} y_{-k}$ ...(2)

Now, putting the values of odd differences in (1) in terms of even difference by (2), we get

$$y_u = y_0 + u\,(y_1 - y_0) + \frac{u^{(2)}}{2\,!}\,\Delta^2 y_{-1} + \frac{(u+1)^{(3)}}{3\,!}\,(\Delta^2 y_0 - \Delta^2 y_{-1}) + \frac{(u+1)^{(4)}}{4\,!}\,\Delta^4 y_{-2}$$

$$+ \ldots + \frac{(u+k-1)^{(2k)}}{(2k)\,!}\,\Delta^{2k} y_{-k} + \frac{(u+k)^{(2k+1)}}{(2k+1)\,!}\,(\Delta^{2k} y_{1-k} - \Delta^{2k} y_{-k} + \ldots$$

[in terms of factorial notation]

$$= (1-u)\; y_0 + u y_1 + \left\{\frac{u^{(2)}}{2\,!} - \frac{(u+1)^{(3)}}{3\,!}\right\}\Delta^2 y_{-1} + \frac{(u+1)^{(3)}}{3\,!}\,\Delta^2 y_0 + \ldots$$

$$+ \left\{\frac{(u+1)^{(4)}}{4\,!} - \frac{(u+2)^{(5)}}{5\,!}\right\}\Delta^4 y_{-2} + \frac{(u+2)^{(5)}}{5\,!}\,\Delta^4 y_{-1} + \ldots$$

$$+ \left\{\frac{(u+k-1)^{(2k)}}{(2k)\,!} - \frac{(u+k)^{(2k+1)}}{(2k+1)\,!}\right\}\Delta^{2k} y_{-k} + \frac{(u+k)^{(2k+1)}}{(2k+1)\,!}\,\Delta^{2k} y_{1-k} + \ldots \qquad \ldots(3)$$

But $$\frac{(u+k-1)^{(2k)}}{(2k)\,!} - \frac{(u+k)^{(2k+1)}}{(2k+1)\,!} = \frac{(u+k-1)^{(2k)}}{(2k+1)\,!}\,\{(2k+1) - (u+k)\}$$

$$= \frac{(k+1-u)\,(u+k-1)^{(2k)}}{(2k+1)\,!} = -\,\frac{(u-k-1)\,(u+k-1)^{(2k)}}{(2k+1)\,!}$$

$$= -\,\frac{(u+k-1)^{(2k+1)}}{(2k+1)\,!} = -\,\frac{(k-v)^{(2k+1)}}{(2k+1)\,!} \qquad [\text{Putting } v = 1-u]$$

$$= -\,\frac{(v+k)^{(2k+1)}}{(2k+1)\,!} = \frac{v\,(v^2-1)\,(v^2-4)\ldots(v^2-k^2)}{(2k+1)\,!}.$$

Now, introducing the variable v, the equation (3) yields

$$y_u = v y_0 + \frac{v\,(v^2-1)}{3\,!}\,\Delta^2 y_{-1} + \frac{v\,(v^2-1)\,(v^2-4)}{5\,!}\,\Delta^4 y_{-2} + \ldots$$

$$+\, v\,\frac{(v^2-1)\,(v^2-4)\ldots(v^2-k^2)}{(2k+1)\,!}\,\Delta^{2k} y_{-k} + \ldots$$

$$+\, u y_1 + \frac{u\,(u^2-1)}{3\,!}\,\Delta^2 y_0 + \frac{u\,(u^2-1)\,(u^2-4)}{5\,!}\,\Delta^4 y_{-1} + \ldots$$

$$+\, \frac{u\,(u^2-1)\,(u^2-4)\ldots(u^2-k^2)}{(2k+1)\,!}\,\Delta^{2k} y_{1-k} + \ldots \qquad \ldots(4)$$

This formula is extensively used and involves only even differences on and below the central line:

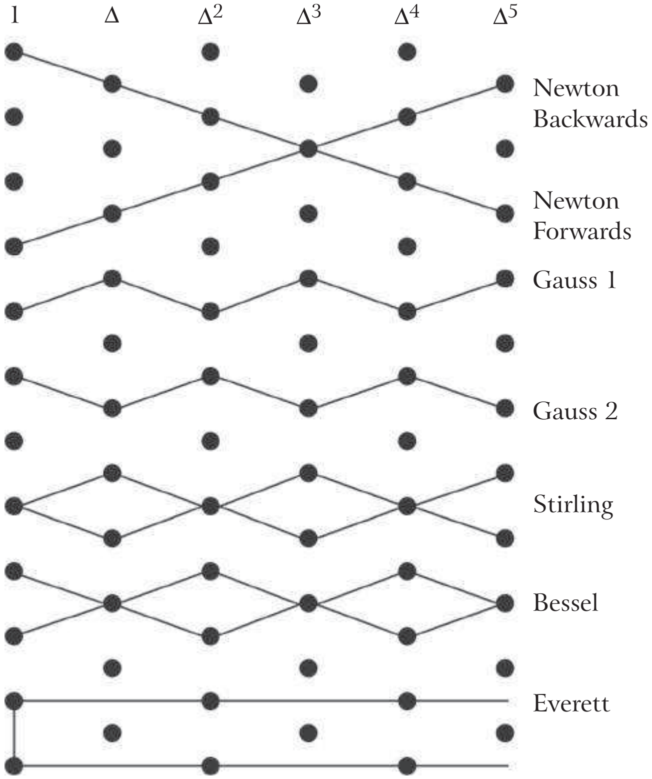
y_0 .... $\Delta^2 y_{-1}$ .... $\Delta^4 y_{-2}$ .... $\Delta^6 y_{-3}$ .... central line

....

y_1 $\Delta^2 y_0$ $\Delta^4 y_{-1}$ $\Delta^6 y_{-2}$

This is ***Laplace-Everett's formula*** in the most common form. This formula is convenient especially when using tables in which only even order differences are tabulated.

Table representing the zigzag paths for various formulae.



SOLVED EXAMPLES

Example 1: ***Prove that***

(*i*) $\Delta \equiv \frac{1}{2}\delta^2 + \delta\left(1 + \frac{\delta^2}{4}\right)^{\frac{1}{2}}$ (*ii*) $\delta^3 y_{1/2} = y_2 - 3y_1 + 3y_0 - y_{-1}$,

[B.C.A. (Rohilkhand) 2005, 2008, 2009, 2011; B.C.A. (Meerut) 2002, 2005, 2007, 2010]

(*iii*) $\Delta\nabla \equiv \nabla\Delta \equiv \Delta - \nabla \equiv \delta^2$ (*iv*) $\delta^n y_x \equiv \Delta^n y_{x-(n/2)}$.

[B.C.A. (Rohilkhand) 2010; B.C.A. (Meerut) 2003]

Solution: (i) $\Delta \equiv \frac{1}{2}\delta^2 + \delta\left(1 + \frac{\delta^2}{4}\right)^{\frac{1}{2}}$

$$= \frac{(E^{1/2} - E^{-1/2})^2}{2} + (E^{1/2} - E^{-1/2})\left(1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}\right)^{\frac{1}{2}}$$

$$= \frac{(E^{1/2} - E^{-1/2})^2}{2} + (E^{1/2} - E^{-1/2})\left\{\frac{4 + (E^{1/2} - E^{-1/2})^2}{4}\right\}^{\frac{1}{2}}$$

$$= \frac{E + E^{-1}}{2} - 1 + \frac{(E - E^{-1})}{2} = E - 1 = \Delta.$$

(ii) $\delta^3 y_{1/2} = (\Delta E^{-1/2})^3 y_{1/2}$ $\qquad [\because \ \delta \equiv \Delta E^{-1/2}]$

$$= \Delta^3 E^{-3/2} E^{1/2} y_0 = \Delta^3 E^{-1} y_0 = (E - 1)^3 E^{-1} y_0 \qquad [\because \ \Delta \equiv E - 1]$$

$$= (E^3 - 3E^2 + 3E - 1) E^{-1} y_0 = (E^2 - 3E + 3 - E^{-1}) y_0 = y_2 - 3y_1 + 3y_0 - y_{-1}.$$

(iii) $\Delta \nabla = (\delta E^{1/2})(\delta E^{-1/2}) = \delta^2$, $\nabla \Delta = (\delta E^{-1/2})(\delta E^{1/2}) = \delta^2$,

and $\qquad \Delta - \nabla = \delta E^{1/2} - \delta E^{-1/2} = \delta(E^{1/2} - E^{-1/2}) = \delta . \delta = \delta^2.$

$$\Delta \nabla \equiv \nabla \Delta \equiv \Delta - \nabla \equiv \delta^2$$

(iv) $\delta^n y_x = (\Delta E^{-1/2})^n y_x = (\Delta^n E^{-n/2})\, y_x = \Delta^n (E^{-n/2} y_x) = \Delta^n y_{x-(n/2)}$.

Example 2: ***Use Gauss's forward formula to find the value of y when $y = 3.75$ from the following table:***

x	2.5	3.0	3.5	4.0	4.5	5.0
y	24.145	22.043	20.225	18.644	17.262	16.047

[B.C.A. (Avadh) 2002, 2004; B.C.A. (Kashi) 2008, 2011]

Solution: Taking 3.5 as the origin and 0.5 as the unit, the value of y required will be value for $u = \dfrac{3.75 - 3.5}{0.5} = 0.5$.

Difference table is:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
2.5	−2	24.145					
			−2.102				
3.0	−1	22.043		.284			
			−1.818		−.047		
3.5	0	20.225		.237		.009	
			−1.581		−.038		−.003
4.0	1	18.644		.199		.006	
			−1.382		−.032		
4.5	2	17.262		.167			
			−1.215				
5.0	3	16.047					

Again Gauss's forward formula is

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_{-1} + \frac{(u+1)\,u\,(u-1)}{6}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)\,u\,(u-1)\,(u-2)}{24}\Delta^4 y_{-2} + \frac{(u+2)\,(u+1)\,(u-1)\,u\,(u-2)}{120}\Delta^5 y_{-2}$$

$$\therefore \qquad y_{.5} = 20.225 + .5\,(-1.581) + \frac{(.5)\,(.5-1)}{2}(.237)$$

$$+ \frac{(.5+1)\,(.5)\,(.5-1)}{6} \times (-.038) + \frac{(.5+1)\,(.5)\,(.5-1)\,(.5-2)}{24} \times (.009)$$

$$+ \frac{(.5+2)\,(.5+1)\,(.5)\,(.5-1)\,(0.5-2)}{120} \times (-.003)$$

$$= 20.225 - 0.7905 - 0.029625 + .00238 + 0.00210938 - 0.000352$$

$$= 19.40 \text{ approx.}$$

Thus value of y for $x = 3.75$ is 19.40 approximately.

Example 3: ***Use Gauss's forward formula to find y_{30} given that***

$y_{21} = 18.4708;\ y_{25} = 17.8144;\ y_{29} = 17.1070;\ y_{33} = 16.3432;\ y_{37} = 15.5154.$

[B.C.A. (Meerut) 2004, 2006; B.B.A. (Meerut) 2007; B.C.A. (Purvanchal) 2005, 2009]

Solution: Arrange the above data in the following form:

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Take the origin at $x = 29$ and $h = 4$ as the unit; then the value of y for $u = \frac{30-29}{4} = 0.25$ is to be found. Construct the following difference table:

x	$u = \frac{x-29}{4}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
21	−2	18.4708				
			−0.6564			
25	−1	17.8144		−0.0510		
			−0.7074		−0.0054	
29	0	17.1070		−0.0564		−0.0022
			−0.7638		−0.0076	
33	1	16.3432		−0.640		
			−0.8278			
37	2	15.5154				

Now applying the Gauss forward interpolation formula, we get

$$y_{0.25} = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)(-0.75)}{2}(-0.0564)$$
$$+ \frac{(1.25)(0.25)(-0.75)}{6}(-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(-0.0022)$$
$$= 17.1070 - .19095 + .0052875 + .00002968 - .00000375 = 16.9216 \text{ approx.}$$

Example 4: ***Given that***

$$\sqrt{(12500)} = 111.803399; \qquad \sqrt{(12510)} = 111.848111;$$
$$\sqrt{(12520)} = 111.892806; \qquad \sqrt{(12530)} = 111.937483.$$

Show by Gauss's backward formula that $\sqrt{(12516)} = 111.874930$.

Solution: Taking 12520 as the origin and $h = 10$ as the unit, the value of y required is for $u = \frac{12516 - 12520}{10} = -0.4$.

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
12500	−2	111.803399			
			0.044712		
12510	−1	111.848111		−0.000017	
			0.044695		−0.000001
12520	0	111.892806		−0.000018	
			0.044677		
12530	1	111.937483			

Now by Gauss's backward formula, we have

$$y_u = y_0 + {}^{u}C_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-2} + \ldots$$

$$\therefore \quad y_{-0.4} = 111.8928 + (-0.4)(0.044695)$$
$$+ \frac{(0.6)(-0.4)}{2!}(-0.000018) + \frac{(0.6)(-0.4)(-1.4)}{3!}(-0.000008)$$
$$= 111.8928 - 0.017878 + \text{negligible quantities} = 111.874930.$$

i.e. $\sqrt{(12516)} = 111.874930$.

Example 5: ***Find by Gauss's backward formula the sales by a concern for the year* 1936*, given:***

Year	1901	1911	1921	1931	1941	1951
Sale (in thousands)	12	15	20	27	39	52

[B.C.A. (Meerut) 2003, 2006]

Solution: Taking 1931 as the origin and $h = 10$ years as the unit, then sale of the concern is to be found for $u = \frac{1936 - 1931}{10} = .5$

The difference table is as under:

x	$u = \frac{x-1931}{10}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1901	−3	12					
			3				
1911	−2	15		2			
			5		0		
1921	−1	20		2		3	
			7		3		−10
1931	0	27		5		−7	
			12		−4		
1941	1	39		1			
			13				
1951	2	52					

Again Gauss backward formula is

$$y_u = y_0 + {}^{u}C_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-2} + \ldots$$

or
$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)\,u}{2}\Delta^2 y_{-1} + \frac{(u+1)\,u\,(u-1)}{6}\Delta^3 y_{-2} + \frac{(u+2)(u+1)\,u\,(u-1)}{24}\Delta^4 y_{-2} + \ldots$$

or
$$y_0 = 27 + 0.5 \times 7 + \frac{(1.5)\times(.5)}{2}\times 5 + \frac{(1.5)(0.5)(-0.5)}{6}\times 3 + \frac{(2.5)(1.5)(0.5)(-0.5)}{24}\times(-7) + \frac{(2.5)(1.5)(0.5)(-0.5)}{120}\times(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2734 - 0.11718$$

$$= 32.6484 - 0.30468 = 32.3437 \text{ thousand.}$$

Example 6: ***Use Stirling's formula to find y_{28}, given***

$y_{20} = 49225,\ y_{25} = 48316,\ y_{30} = 47236,\ y_{35} = 45926,\ y_{40} = 44306.$

[B.C.A. (Bhopal) 2002, 2005, 2012; B.C.A. (Avadh) 2004, 2005, 2008]

Solution: Taking $x = 30$ as the origin and $h = 5$ as the unit, we have to find the value of y for $u = \frac{28-30}{5} = -0.4$.

Construct the following difference table:

x	$u = \frac{x-30}{5}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
20	−2	49225				
			−909			
25	−1	48316		−171		
			−1080		−59	
30	0	47236		−230		−21
			−1310		−80	
35	1	45926		−310		
			−1620			
40	2	44306				

The Stirling's formula is

$$y_u = y_0 + u\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{6}\cdot\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{u^2(u^2-1)}{24}\Delta^4 y_{-2}.$$

Putting $u = -0.4$ and the values of various differences from the table, we get

$$y_{-0.4} = 47236 + (-0.4)\frac{-1310-1080}{2} + \frac{(0.16)}{2}(-230) + \frac{(-0.4)(0.16-1)}{6}\cdot\frac{(-80-59)}{2} + \frac{(0.16)(0.16-1)}{24}(-21)$$

$$= 47236 + 478 - 18.4 - 3.8920 + .1176 = 47692 \quad i.e.\ y_{28} = 47692.$$

Example 7: ***Apply Bessel's formula to obtain y_{25}, given***

$$\boldsymbol{y_{20} = 2854,\ y_{24} = 3162,\ y_{28} = 3544,\ y_{32} = 3992.}$$

[B.C.A. (Agra) 2006, 2008]

Solution: Take 24 as the origin and 4 as the unit. We have to find y_u for value of $u = \frac{25-24}{4} = \frac{1}{4}$ or 0.25.

The difference table is:

x	$u = \frac{x-24}{4}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	−1	2854			
			308		
24	0	3162		74	
			382		−8
28	1	3544		66	
			448		
32	2	3992			

Bessel's formula is

$$y_u = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2\,!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3\,!}\Delta^3 y_{-1} + \dots$$

$$\therefore y_{1/4} = \frac{1}{2}(3162 + 3544) + \left(\frac{1}{4} - \frac{1}{2}\right)382 + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4} - 1\right)}{2\,!}\cdot\left(\frac{74+66}{2}\right) + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4} - 1\right)}{3\,!}(-8)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625 = 3250.875.$$

Example 8: ***Given*** $y_{20} = 24,\ y_{24} = 32,\ y_{28} = 35,\ y_{32} = 40$, ***find*** y_{25} ***by Bessel's formula.*** **[B.C.A. (Kanpur) 2008]**

Solution: Take origin at $x = 24$ and 4 as the unit, we have to find y_u for $u = \frac{25-24}{4} = \frac{1}{4}$.

The difference table is as under:

x	$u = \frac{x-24}{4}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	−1	24			
			8		
24	0	32		−5	
			3		7
28	1	35		2	
			5		
32	2	40			

The Bessel's formula is

$$y_u = \frac{y_0 + y_{-1}}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3\,!}\Delta^3 y_{-1}$$

Putting the values of various differences so obtained and $u = \frac{1}{4}$ we get

$$y_{1/4} = \frac{32+34}{2} + \left(\frac{1}{4} - \frac{1}{2}\right)3 + \frac{\frac{1}{4}\left(\frac{1}{4} - 1\right)}{2}\cdot\left(\frac{-5+2}{2}\right) + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4} - 1\right)}{6}(7)$$

$$= 33.5 - 0.75 + 0.1406 + 0.054687 = 32.945287.$$

Example 9: ***Apply Everett's formula to obtain*** y_{25}, ***given***

$$y_{20} = 2854,\ y_{24} = 3162,\ y_{28} = 3544,\ y_{32} = 3992.$$

Solution: Take $x = 24$ as the origin and $h = 4$ as the unit, we have to find y_u for $u = \frac{25-24}{4} = \frac{1}{4}$ or 0.25.

The difference table is as under:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

The Everett's formula is

$$y_u = uy_1 + \frac{u(u^2-1)}{3\,!}\Delta^2 y_0 + \frac{u(u^2-1)(u^2-4)}{5\,!}\Delta^4 y_{-1} + \ldots$$

$$+ vy_0 + \frac{v(v^2-1)}{3\,!}\Delta^2 y_{-1} + \ldots \quad \text{where } v = 1-u$$

$$= \frac{1}{4}(3544) + \frac{\frac{1}{4}\left(\frac{1}{16}-1\right)}{6} \times 66 + \frac{3}{4}(3162) \times \frac{\frac{3}{4}\left(\frac{9}{16}-1\right)}{6} \times 74$$

$$= 886 - 2.5781 + 2371.5 - 4.0469 = 3250.875\,.$$

Example 10: ***From the following table, find the value of* log 337.5 *by Gauss, Stirling, Bessel and Everett formulae.***

x	310	320	330	340	350	360
$\log_{10} x$	2.4913617	2.5051500	2.5185139	2.5314789	2.5440680	2.5563025

Solution: Taking 330 as the origin and 10 as the unit, the value of $\log_{10}$ 337.5 required will be for $u = \dfrac{337.5-330}{10} = .75$.

The difference table is as under:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
310	-2	2.4913617					
			.0137883				
320	-1	2.5051500		$-.0004244$			
			.0133639		.0000255		
330	0	2.5185139		$-.0003989$		-25×10^{-7}	
			.0129650		.0000230		8×10^{-7}
340	1	2.5314789		$-.0003759$		-17×10^{-7}	
			.0125891		.0000213		
350	2	2.5440680		$-.0003546$			
			.0122345				
360	3	2.5563025					

(i) Gauss's forward formula is

$$y_u = y_0 + {}^{u}C_1 \Delta y_0 + {}^{u}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^{u+1}C_4 \Delta^4 y_{-2} + {}^{u+2}C_5 \Delta^5 y_{-2} + \ldots$$

$$\therefore \quad y_{.75} = 2.5185139 + \frac{3}{4}(.0129650) + \frac{\frac{3}{4}.\left(-\frac{1}{4}\right)}{2}(-.0003989)$$

$$+ \frac{\frac{7}{4}.\frac{3}{4}\left(-\frac{1}{4}\right)}{6}(.0000230) + \frac{\frac{7}{4}.\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{24}(-.0000025)$$

$$+ \frac{\frac{11}{4}.\frac{7}{4}.\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{120}(.0000008) = 2.52827374.$$

(ii) Gauss's backward formula is

$$y_u = y_0 + {}^{u}C_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-2} + {}^{u+1}C_4 \Delta^4 y_{-2} + \ldots$$

$$= 2.5185139 + \frac{3}{4}(.0133639) + \frac{\frac{7}{4}.\frac{3}{4}}{2}(.0003989)$$

$$+ \frac{\frac{7}{4}.\frac{3}{4}\left(-\frac{1}{4}\right)}{6}(.0000255) + \frac{\frac{11}{4}.\frac{7}{4}.\frac{3}{4}.\left(-\frac{1}{4}\right)}{24}(.0000025) = 2.52827375.$$

(iii) Stirling's formula is $y_u = y_0 + u\,.\,\frac{1}{2}[\Delta y_0 + \Delta y_{-1}] + \frac{u^2}{2}\Delta^2 y_{-1}$

$$+ \frac{u(u^2-1)}{(3)!}\,\frac{1}{2}[\Delta^3 y_{-1} + \Delta^3 y_{-2}] + \frac{u^2(u^2-1)}{(4)!}\Delta^4 y_{-2} + \ldots$$

$$= 2.5185139 + \frac{3}{4}.\frac{1}{2}[.0129650 + .0133639] + \frac{\left(\frac{3}{4}\right)^2}{2}(.0003989)$$

$$+ \frac{\frac{3}{4}\left(\frac{9}{16}-1\right)}{6}\,\frac{1}{2}.[.0000230 + .000255] + \frac{\frac{9}{16}\left(\frac{9}{16}-1\right)}{24}(.0000025) = 2.52827374.$$

(iv) Bassel's formula is $y_u = \frac{1}{2}[y_0 + y_1] + \left(u - \frac{1}{2}\right)\Delta y_0$

$$+ \frac{u(u-1)}{(2)!}\,\frac{1}{2}[\Delta^2 y_{-1} + \Delta^2 y_0] + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)\,u\,(u-1)\,(u-2)}{4!}\,\frac{1}{2}[\Delta^4 y_{-1} + \Delta^4 y_{-2}]$$

$$+ \frac{\left(u-\frac{1}{2}\right)(u+1)\,u\,(u-1)\,(u-2)}{5!}\Delta^5 y_{-2} + \ldots$$

$$= \frac{1}{2}[0.5185139 - 2.5314789] + \frac{1}{4}(.0129650)$$

$$+ \frac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2}\,\frac{1}{2}[-.0003989 - .0003759] + \frac{\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{6}(.0000230)$$

$$+ \frac{\frac{7}{4}.\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{24}\,\frac{1}{2}[-.0000017 - .0000025]$$

$$+ \left(\frac{11}{4}\right)\left(\frac{7}{4}\right)\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)(.0000008) = 2.52827374.$$

(v) Everett's formula is

$$y_u = uy_1 + \frac{u(u^2-1)}{3!}\Delta^2 y_0 + \frac{u(u^2-1)(u^2-4)}{5!}\Delta^4 y_{-1} + \ldots$$

$$+ vy_0 + \frac{v(v^2-1)}{(3)!}\Delta^2 y_{-1} + \frac{v(v^2-1)(v^2-4)}{5!}\Delta^4 y_{-2} + \ldots \text{ where } v = 1-u$$

$$= \frac{3}{4}(2.5314789) + \frac{\frac{3}{4}\left(\frac{9}{16}-1\right)}{6}(-.0003759)$$

$$+ \frac{\frac{3}{4}\left(\frac{9}{16}-1\right)\left(\frac{9}{16}-4\right)}{120}(-.0000017) + \frac{1}{4}(2.5185139)$$

$$+ \frac{\frac{3}{4}\left(\frac{1}{16}-1\right)}{6}(-.0003989) + \frac{\frac{1}{4}\left(\frac{1}{16}-1\right)\left(\frac{1}{16}-4\right)}{120}(.0000025)$$

$$= 2.52827374.$$

Thus we see that the value of log 337.5 calculated from different formula is nearly the same.

Example 11:* *Given:

θ	0°	5°	10°	15°	20°	25°	30°
tan θ	0.0000	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

***Show that* tan 16° = 0.2867 (*use Stirling's formula*).**

Solution: Taking 15° as the origin and 5° as the unit, the value (of tan 16°) required will be for $u = \frac{16° - 15°}{5°} = .2°$.

The difference table is as under:

θ	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$	$\Delta^6 y_u$
0°	−3	0.0000						
			.0875					
5°	−2	0.0875		.0013				
			.0888		.0015			
10°	−1	0.1763		.0028		.0002		
			.0916		.0017		−.0002	
15°	0	0.2679		.0045		.0000		.0011
			.0961		.0017		.0009	
20°	1	0.3640		.0062		.0009		
			.1023		.0026			
25°	2	0.4663		.0088				
			.1111					
30°	3	0.5774						

Stirling's formula is

$$y_u = y_0 + u\frac{1}{2}[\Delta y_0 + \Delta y_{-1}] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{1}{2}[\Delta^3 y_{-1} + \Delta^3 y_{-2}]$$

$$+ \frac{u^2(u^2-1)}{(4)!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{(5)!}\frac{1}{2}[\Delta^5 y_{-2} + \Delta^5 y_{-3}]$$

$$+ \frac{u^2(u^2-1)(u^2-4)}{(6)!}\Delta^6 y_{-3}$$

Now putting the necessary values from the difference table, we have

$$\tan 16° = y_{.2} = 0.2679 + \frac{.2}{2}[.0961 + .0916] + \frac{.04}{2}(.0045)$$

$$+ \frac{.2(.04-1)}{6}\frac{1}{2}(.0017 + .0017) + \frac{.04(.04-1)}{24}.0000$$

$$+ \frac{.2(.04-1)(.04-4)}{120}\frac{1}{2}[-.0002 + .0009] + \frac{.2(.04-1)(.04-4)}{720}(.0011)$$

$$= .2679 + .01877 + .00009 + \text{negligible quantities} = 0.2867.$$

Example 12: ***If third differences are constant, prove that***

$$y_{x+1/2} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x).$$

Solution: Bessel's formula is

$$y_x = \frac{y_0 + y_1}{2} + \left(x - \frac{1}{2}\right)\Delta y_0 + \frac{x(x-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(x - \frac{1}{2}\right)x(x-1)}{3!}\Delta^3 y_{-1}. \quad \text{...(1)}$$

Formula is taken upto third differences since third differences are constant.

Putting $x = \frac{1}{2}$ in (1), we get

$$y_{1/2} = \frac{y_0 + y_1}{2} - \frac{1}{16}(\Delta^2 y_{-1} + \Delta^2 y_0) \quad \text{...(2)}$$

Now shift the origin to x; thus (2) reduces to

$$y_{x+1/2} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x+1} + \Delta^2 y_x).$$

Example 13: ***Given $y_0, y_1, y_2, y_3, y_4, y_5$ (Fifth differences constant). Prove that***

$$y_{\left(2\frac{1}{2}\right)} = \frac{1}{2}c + \frac{25(c-b) + 3(a-c)}{256}$$

where $a = y_0 + y_5;\ b = y_1 + y_4;\ c = y_2 + y_3.$

[B.C.A. (Kanpur) 2001, 2004, 2006; B.C.A. (Kurukshetra) 2010]

Solution: Putting $x = \frac{1}{2}$ in the Bessel's formula, we get

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{8}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3}{128}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) \quad \text{...(1)}$$

We have written the formula upto fifth differences, since it is given that fifth differences are constant.

Now shift the origin to 2, so (1) reduces to

$$y_{\left(2\frac{1}{2}\right)} = \frac{1}{2}(y_2 + y_3) - \frac{1}{16}(\Delta^2 y_1 + \Delta^2 y_2) + \frac{3}{256}(\Delta^4 y_0 + \Delta^4 y_1) \quad \text{...(2)}$$

Now consider the following difference table:

y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
y_0				
	$y_1 - y_0$			
y_1		$y_2 - 2y_1 + y_0$		
	$y_2 - y_1$		$y_3 - 3y_2 + 3y_1 - y_0$	
y_2		$y_3 - 2y_2 + y_1$		$y_4 - 4y_3 + 6y_2 - 4y_1 - y_0$
	$y_3 - y_2$		$y_4 - 3y_3 + 3y_2 - y_1$	
y_3		$y_4 - 2y_3 + y_2$		$y_5 - 4y_4 + 6y_3 - 4y_2 - y_1$
	$y_4 - y_3$		$y_5 - 3y_4 + 3y_3 - y_2$	
y_4		$y_5 - 2y_4 + y_3$		
	$y_5 - y_4$			
y_5				

Now putting the values of various differences from the table in (2), we get

$$y_{\left(2\frac{1}{2}\right)} = \frac{1}{2}(y_2 + y_3) - \frac{1}{16}(y_4 - y_3 - y_2 + y_1)$$

$$+ \frac{3}{256}(y_5 - 3y_4 + 2y_3 + 2y_2 - 3y_1 + y_0)$$

or

$$y_{\left(2\frac{1}{2}\right)} = \frac{1}{2}c - \frac{1}{16}(b - c) + \frac{3}{256}(a - 3b + 2c)$$

$$= \frac{1}{2}c + \frac{1}{256}[16\,(c - b) + 3\,\{(a - c) + 3\,(c - b)\}]$$

$$= \frac{1}{2}c + \frac{1}{256}[3\,(a - c) + 25\,(c - b)].$$

Example 14: ***Employ Stirling's formula to compute*** $y_{12.2}$ ***from the following table*** **(**$y_x = 1 + \log_{10} \sin x$**):**

x	10	11	12	13	14
$10^5 u_x$	23,967	28,060	31,788	35,209	38,368

Solution: Taking the origin at $x = 12, h = 1$ and $u = \dfrac{x - 12}{1}$, we have the following central table:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
10	−2	.23967				
			.04093			
11	−1	.28060		−.00365		
			.3728		**.00058**	
12	0	**.31788**		**−.00307**		**−.00013**
			.03421		**−0.00045**	
13	1	.35209		−.00062		
			.03159			
14	2	.38368				

At $x = 12.2$, $u = 0.2$. [$\because u$ lies between $-\dfrac{1}{4}$ and $\dfrac{1}{4}$, the use of Stirling's formula is suitable.]

Now Stirling's formula is given by

$$y_u = y_0 + \frac{u}{1} \cdot \frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{u^2}{2\,!} \Delta^2 y_{-1} + \frac{u(u^2 - 1)}{3\,!} \times \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}$$

$$\times \frac{u^2 (u^2 - 1)}{4\,!} \Delta^4 y_{-2} + \ldots$$

When $u = 0.2$, we readily get

$$\therefore \qquad y_{0.2} = 0.31788 + 0.2 \left(\frac{0.03728 + 0.03421}{2} \right) + \frac{(0.2)^2}{2} (-0.00307)$$

$$+ \frac{(0.2)\,[(0.2)^2 - 1]}{6} \left(\frac{0.00058 - 0.00045}{2} \right)$$

$$+ \frac{(0.2)^2\,[(0.2)^2 - 1]}{24} (-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002$$

$$= 0.32497.$$

Problem Set

1. Use Gauss's interpolation formula to find y_{41} with the help of following data:

   $y_{30} = 3678.2$; $y_{35} = 2995.1$; $y_{40} = 2400.1$; $y_{45} = 1876.2$; $y_{50} = 1416.3$.

   [B.C.A. (Lucknow) 2004, 2011; (Meerut) 2008]

2. Apply a central difference formula to obtain y_{22}, given that $y_{25} = 0.2707$; $y_{30} = 0.3027$; $y_{35} = 0.3386$; $y_{40} = 0.3794$.

3. Given $y_2 = 10$; $y_1 = 8$; $y_0 = 5$; $y_{-1} = 10$; find $y_{1/2}$ by Gauss's forward formula table. Find the value of e^{-x} when $x = 1.7489$ by Gauss, Strilling, Bessel and Everett formulae.

x	e^{-x}
1.72	.179066
1.73	.177284
1.74	.175520
1.75	.173774
1.76	.172045
1.77	.170333
1.78	.168638

[B.C.A. (Agra) 2003]

4. The mean atmospheric refraction R for a start at various altitudes $h°$ above the horizon is given in the table below. Using Bessel's formula for interpolation to halves, find the refraction for a star at an altitude of 27° above the horizon:

h	22°	24°	26°	28°	30°	32°
R	2′23.3″	2′10.2″	1′58.9″	1′49.2″	1′40.6″	1′33.0″

5. Prove that if third differences are assumed to be constant

$$y_x = xy_1 + \frac{x(x^2-1)}{3!}\Delta^2 y_0 + uy_0 + \frac{u(u^2-1)}{3!}\Delta^2 y_{-1}$$

   where $u = 1 - x$.

   Apply this formula to find the value of y_{25} and y_{10} given that $y_0 = 3010$, $y_5 = 2710$, $y_{10} = 2285$, $y_{15} = 1860$, $y_{20} = 1560$, $y_{25} = 1510$ and $y_{30} = 1885$.

   [**Hint:** This formula can be easily proved with the help of Everett's formula].

6. Use Stirling's formula to find $f(35)$ given

$$f(20) = 512,\ f(30) = 439,\ f(40) = 346 \text{ and } f(53) = 243$$

[B.C.A. (Rohilkhand) 2003, 2007]

7. Use Stirling's formula to find $f(1.22)$ from the following data:

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	0.84147	0.89121	0.93204	0.96356	0.98545
x	1.5	1.6	1.7	1.8	
$f(x)$	0.99749	0.99957	0.97385	0.97385	

8. From the following table find the value of $f(.5437)$ by Gauss, Stirling, Bessel and Everett formula: **[B.C.A. (Indore) 2012]**

x	$f(x)$
.51	.529244
.52	.537895
.53	.546464
.54	.554939
.55	.563323
.56	.571616
.57	.579816

9. (i) Prove that: (a) $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$

(b) $\mu^2 = 1 + \frac{1}{4}\delta^2$ **[B.C.A. (Meerut) 2006]**

(c) $E^{\pm 1/2} = \mu \pm \frac{1}{2}\delta$ **[B.C.A. (Meerut) 2005]**

(d) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

(e) $\sqrt{(1 + \delta^2 \mu^2)} = 1 + \frac{1}{2}\delta^2$ **[B.C.A. (Lucknow) 2006, 2010]**

(ii) Establish the following relations:

(a) $\Delta \nabla \equiv \nabla \Delta \equiv \Delta - \nabla \equiv \delta^2$

(b) $\mu\delta \equiv \frac{1}{2}(\nabla + \Delta) \equiv \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

(c) $\Delta + \nabla \equiv \Delta / \nabla - \nabla / \Delta$. **[B.C.A. (Meerut) 2003; B.C.A. (Avadh) 2007]**

10. Prove that (i) $\delta^2 y_0 = y_1 - 2y_0 + y_{-1}$

(ii) $\delta^3 y_{1/2} = y_2 - 3y_1 + 3y_0 - y_{-1}$, and in general

$$\delta^n y_x = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} y_{x+1/2n-k}$$

[**Hint:** $\delta^n = E^{-n/2}(E-1)^n \quad \therefore \quad \delta^n y_x = E^{-n/2}(E-1)^n y_x = (E-1)^n y_{x-n/2}$ etc.]

11. Define the operator E, Δ, μ and δ used in the calculus of finite differences and prove the following relations:

(i) $\delta^n y_x = \Delta^n y_{x-n/2}$ (ii) $\mu^2 y_x = y_x + \frac{1}{4}\delta^2 y_x$ (iii) $E^{1/2} y_x = \mu y_x + \frac{1}{2}\delta y_x$.

12. If D, E, δ and μ be the operators with usual meanings and if $hD \equiv U$, where h is the interval of differencing prove the following relations between the operator.

(i) $E = e^U$

(ii) $\delta = 2 \sin h (U/2)$

(iii) $\mu = \cos h (U/2)$

(iv) $e^{-U} = 1 - \nabla$

(v) $(E+1)\delta = 2(E-1)\mu$

(vi) $\delta[f(x)\,g(x)] = \mu\, f(x)\,\delta\, g(x) + \mu\, g(x)\,\delta\, f(x)$

Answers

1.	$y_{41} = 2290.1$	2.	.3165
3.	$y_{1/2} = 6$	4.	1′52.4″
5.	$y_{11} = 2196$, $y_{16} = 1786$	6.	395
7.	.93258	8.	.558052

❍❍❍

Unit

3

Chapter-6: Numerical Differentiation

Chapter-7: Numerical Integration

Unit-3

Chapter 6

Numerical Differentiation

6.1 Introduction

Numerical differentiation is the process by which we can find the derivative or derivatives of a function at some values of the independent variable when we are given a set of values of that function. The problem of differentiation is solved by first approximating the function by an interpolation formula and then differentiating this formula as many times as desired. In case the values of the argument are equally spaced, we represent the function by Newton-Gregory formula. If we desire to find the derivative of the function at a point near the beginning (end) of a set of tabular values, we use Newton-Gregory forward (backward) formula. To find the derivative at a point near the middle of the table, we should use a central difference formula. In case the values of the argument are unequally spaced, we should use ***Newton's divided difference*** formula to represent the function.

SOLVED EXAMPLES

Example 1: ***Find the first and second derivatives of the function tabulated below at the point*** $x = 3.0$:

[B.C.A. (Lucknow) 2001, 2005; B.C.A. (Avadh) 2010; B.C.A. (Bhopal) 2006, 2010; B.C.A. (Kashi) 2012]

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
3.0	−14.000				
		3.968			
3.2	−10.032		.768		
		4.736		.048	
3.4	−5.296		.816		0
		5.552		.048	
3.6	0.256		.864		0
		6.416		.048	
3.8	6.672		.912		
		7.328			
4.0	14.000				

Solution: Since the derivatives are required at $x=3$ which is near the beginning of the table, so we use Newton-Gregory formula *i.e.*

$$f(a+xh) = f(a) + {}^xC_1 \Delta f(a) + {}^xC_2 \Delta^2 f(a) + {}^xC_3 \Delta^3 f(a)$$

$$= f(a) + x\Delta\, f(a) + \frac{x^2 - x}{2} \Delta^2 f(a) + \frac{(x^3 - 3x^2 + 2x)}{6} \Delta^3 f(a)$$

Differentiating w.r.t. x twice and then putting $x=0$ in the equations obtained, we get

$$hf'(a) = \Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a)$$

$$h^2 f''(a) = \Delta^2 f(a) - \Delta^3 f(a).$$ Here $a=3, h=0.2$.

$\therefore$ $0.2 f'(3) = 3.968 - \frac{1}{2}(.768) + \frac{1}{5}(.048) = 3.6.$ $\therefore$ $f'(3) = 18.$

Again $(0.2)^2\ f''(3) = .768 - .048 = .72$

$\therefore$ $f''(3) = 18,$ $\therefore$ $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=3$ are 18.

Note: The function which is tabulated above is

$$y = x^3 - 9x - 14 \qquad \frac{dy}{dx} = 3x^2 - 9;\ \frac{d^2y}{dx^2} = 6x.$$

On putting $x=3$, we get $\left(\frac{dy}{dx}\right)_{x=3} = 18;\ \left(\frac{d^2y}{dx^2}\right)_{x=3} = 18.$

These values are the same as obtained by numerical differentiation.

Example 2: ***Find the first, second and third derivatives of the function tabulated below, at the point*** $x = 1.5$.

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

[B.C.A. (Avadh) 2003]

Solution: Since required derivatives are at a point near the beginning of the table, therefore we shall use Newton-Gregory forward formula. The difference table is as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		.750	
2.5	13.625		3.750		0
		10.375		.750	
3.0	24.000		4.500		0
		14.875		.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Newton-Gregory formula is

$$f(a+xh) = f(a) + {}^xC_1\Delta f(a) + {}^xC_2\Delta^2 f(a) + {}^xC_3\Delta^3 f(a) \text{ upto 3rd differences}$$

$$= f(a) + x\Delta f(a) + \frac{x^2 - x}{2}\Delta^2 f(a) + \frac{(x^3 - 3x^2 + 2x)}{6}\Delta^3 f(a)$$

Differentiating w.r.t. x thrice and then putting $x = 0$ in the equations obtained, we get

$$hf'(a) = \Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a)$$

$$h^2 f''(a) = \Delta^2 f(a) - \Delta^3 f(a),\ h^3 f'''(a) = \Delta^3 f(a). \quad [a = 1.5, h = .5]$$

$$\therefore \qquad f'(1.5) = \frac{1}{.5}\left[3.625 - \frac{1}{2}(3.000) + \frac{1}{3}(.750)\right] = 4.750$$

$$f''(1.5) = \frac{1}{.25}[3.000 - .750] = 9.000.\ f'''(1.5) = \frac{1}{.5^3}[.750] = 6.000.$$

Note: The function which is tabulated above is $y = x^3 - 2x + 3$.

Clearly $\quad (dy/dx)_{1.5} = (3x^2 - 2)$ at $1.5 = 4.750$

$(d^2y/dx^2)_{1.5} = [6x]$ at $1.5 = 9.000$. $\qquad (d^3y/dx^3)_{1.5} = 6.000$.

These values are exactly the same as obtained by numerical differentiation.

Example 3: ***Find the first and second derivatives of the function tabulated below at the point $x = 1.1$:***

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0				
		.1280			
1.2	.1280		.2880		
		.4160		.0480	
1.4	.5440		.3360		0
		.7520		.0480	
1.6	1.2960		.3840		0
		1.1360		.0480	
1.8	2.4320		.4320		
		1.5680			
2.0	4.0000				

[B.C.A. (Agra) 2012]

Solution: Newton's-Gregory formula is

$$f(a+xh) = f(a) + {}^{x}C_1\Delta f(a) + {}^{x}C_2\Delta^2 f(a) + {}^{x}C_3\Delta^3 f(a)\ldots$$

$$= f(a) + x\,\Delta\, f(a) + \frac{x^2-x}{2}\Delta^2 f(a) + \frac{(x^3-3x^2+2x)}{6}\Delta^3 f(a) + \ldots \qquad \ldots(1)$$

Differentiating (1) w.r.t. x twice, we get

$$hf'(a+xh) = \Delta f(a) + \frac{(2x-1)}{2}\Delta^2 f(a) + \frac{(3x^2-6x+2)}{6}\Delta^3 f(a) + \ldots \qquad \ldots.(2)$$

and $$h^2 f''(a+xh) = \Delta^2 f(a) + (x-1)\,\Delta^3 f(a) + \ldots \qquad \ldots(3)$$

Putting $a = 1$, $h = .2$, $x = \frac{1}{2}$ and the values of all differences in (2) and (3), we have

$$(.2)\ f'(1.1) = 0.1280 + \frac{\left(2\,.\,\frac{1}{2}-1\right)}{2}(.2880) + \frac{\left(3\,.\,\frac{1}{4}-6\,.\,\frac{1}{2}+2\right)}{6}(.0480) + 0$$

$$= .1280 + 0 - .002 + 0 \quad \text{or} \quad f'(1.1) = .630$$

and $(.2)^2\ f''(1.1) = .2880 + \left(\frac{1}{2}-1\right)(.0480) + 0 = .2880 - .024 = .264 \quad \text{or} \quad f''(1.1) = 6.60.$

Note: The function, which is tabulated above, is

$$y = x^3 - 3x + 2, \qquad \therefore \quad \frac{dy}{dx} = 3x^2 - 3 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x.$$

Hence the actual values of derivatives are .630 and 6.60 respectively, which are same as obtained by Numerical differentiation.

Example 4: ***Find the derivative of $f(x)$ at $x = 1.4$ from the following table:***

x	.1	.2	.3	.4
$y = f(x)$	1.10517	1.22140	1.34986	1.49182

Solution: Since $x = .4$ lies near the end of the table therefore in this case we shall use Newton's Backward formula. The difference table is as below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
.1	1.10517			
		.11623		
.2	1.22140		.01223	
		.12846		.00127
.3	1.34986		.01350	
		.14196		
.4	1.49182			

Newton's Backward formula is $f(a+nh+xh) = f(a+nh) + x\nabla f(a+nh) + \frac{x^2+x}{2\,!}\nabla^2 f(a+nh)$

$$+\frac{x^3+3x^2+2x}{3\,!}\nabla^3 f(a+nh) \qquad ...(1)$$

Here $f(a+nh)$ is the last term and h is the differencing interval.

Differentiating (1) w.r.t. x, we get

$$hf'(a+nh+xh) = \nabla f(a+nh) + \frac{2x+1}{2}\nabla^2 f(a+nh) + \frac{3x^2+6x+2}{6}\nabla^3 f(a+nh). \qquad ...(2)$$

On putting $x = 0, a+nh = .4, h = .1$ in (2), we have

$$(.1)\, f'(.4) = .14196 + \frac{1}{2}(.01350) + \frac{1}{3}(.00127) \Rightarrow f'(.4) = 1.4913$$

6.2 Direct Methods (using formula)

1. Derivatives using forward difference formula

By Newton's forward interpolation formula, we have

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots$$

where the function $y = f(x)$ is tabulated for $x_i\ (= x_0 + ih), i = 0, 1, 2, \ldots n$

Differentiating both sides w.r.t. u, we have

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2\,!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3\,!}\Delta^3 y_0 + \ldots$$

But $$x = x_0 + uh \Rightarrow u = \frac{(x-x_0)}{h}, \Rightarrow \frac{du}{dx} = \frac{1}{h}.$$

Now $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2\,!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3\,!}\times\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4\,!}\Delta^4 y_0 + \ldots\right] \qquad ...(1)$$

When $x = x_0, u = 0.$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \qquad ...(2)$$

Further differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} = \frac{1}{h}\left[\frac{2}{2\,!}\Delta^2 y_0 + \frac{6u-6}{3\,!}\Delta^3 y_0 + \frac{12u^2-36u+22}{4\,!}\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

Putting $u = 0$, we obtain

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right] \quad \ldots(3)$$

and similarly $\left(\frac{d^3 y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$ ...(4)

Aliter: We know that $1 + \Delta \equiv E \equiv e^{hD}$

$$\Rightarrow \qquad hD = \log(1 + \Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

$$\Rightarrow \qquad D \equiv \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

$$\Rightarrow \qquad D^2 \equiv \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots\right] \text{ and } D^3 \equiv \frac{1}{h^3}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right]$$

Then Dy_0 *i.e.* $\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right]$

$$D^2 y_0 = D(Dy_0) = \left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\right]$$

and $\left(\frac{d^3 y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$

These are the same as obtained above.

2. Derivatives using backward difference formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^2 y_n + \ldots$$

Then differentiating both sides w.r.t. u, we get

$$\frac{dy}{dx} = \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2 + 6u + 2}{3!}\nabla^3 y_n + \ldots$$

But $u = \frac{x - x_n}{h}, \ \Rightarrow \frac{du}{dx} = \frac{1}{h}.$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^3 + 6u + 2}{3!}\nabla^2 y_n + \ldots\right]$ ...(5)

But at $x = x_n$ $u = 0$. Hence keeping $u = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad ...(6)$$

and
$$\frac{d^2 y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx} = \frac{1}{h}\left[\nabla^2 y_n + \frac{6u+6}{3\,!}\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \ldots\right]$$

Then putting $u = 0$, we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad ...(7)$$

on the same lines, we get

$$\left(\frac{d^3 y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right] \quad ...(8)$$

Aliter. $1 - \nabla \equiv E^{-1} \equiv e^{-hD}$, Taking log on both sides

$$\Rightarrow \qquad -hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$\Rightarrow \qquad D \equiv \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right]$$

$$\Rightarrow \qquad D^2 \equiv \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^4 + \ldots\right]^2$$

$$\equiv \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right] \quad \Rightarrow D^3 \equiv \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \ldots\right]$$

Then $\quad Dy_n$ *i.e.* $\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right]$

$$\Rightarrow \qquad \left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right]$$

and
$$\left(\frac{d^3 y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

There are again the same as (6), (7) and (8).

3. Derivatives using central difference formulae

By Stirling's formula, we have

$$y_u = y_0 + \frac{u}{1\,!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2\,!}\Delta^2 y_{-1}$$
$$+ \frac{u(u^2-1^2)}{3\,!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4\,!}\Delta^4 y_{-2} + \ldots$$

$$\Rightarrow \quad \frac{dy}{du} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2u}{2\,!}\Delta^2 y_{-1} + \frac{3u^2 - 1}{3\,!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3 - 2u}{4\,!}\Delta^4 y_{-2} + \ldots$$

But $\qquad u = \frac{x - x_0}{h}, \ \Rightarrow \frac{du}{dx} = \frac{1}{h}.$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\,\Delta^2 y_{-1} + \frac{3u^2 - 1}{6}\times\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2u^3 - u}{12}\Delta^4 y_{-2} + \ldots\right]$$

But at $x = x_0, u = 0$. Hence putting $u = 0$, we have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\right] \qquad \ldots(9)$$

and similarly $\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right]$ ...(10)

Example 5:* *Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ ***and*** $\frac{d^2 y}{dx^2}$ ***at*** ***(i)*** $x = 1.1$ ***(ii)*** $x = 1.6$

[B.C.A. (Purvanchal) 2007, 2010; B.C.A. (Rohtak) 2005, 2006, 2010, 2011]

Solution: (i) The difference table is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		.414					
1.1	8.403		–.036				
		.378		.006			
1.2	8.781		–.030		–.002		
		.348		.004		.002	
1.3	9.129		–.026		.000		–.003
		.322		.004		–.001	
1.4	9.451		–.023		–.001		
		.299		.005			
1.5	9.750		–.018				
		.281					
1.6	10.031						

By the standard formula, we have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad \ldots(1)$$

and $$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{37}{180}\Delta^6 y_0 - \ldots\right] \quad \ldots(2)$$

Here $h = 0.1, x_0 = 1.1, \Delta y_0 = 0.378, \Delta^2 y_0 = -0.03$ etc.

Then putting these values in (1) and (2), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001) - \frac{1}{6}(-0.003)\right] = 3.946$$

and $$\left(\frac{d^2 y}{dx^2}\right)_{1.1} = \frac{1}{(.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right] = -3.545.$$

(ii) Using the above difference table and the backward difference operator ∇ instead of Δ; we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \ldots\right] \quad \ldots(3)$$

and $$\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right] \quad \ldots(4)$$

Here $h = 0.1, x_n = 1.6, \nabla y_n = 0.281, \nabla^2 y_n = -0.018$ etc.

Then putting these values in (3) and (4), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001) + \frac{1}{6}(-0.003)\right] = 2.727$$

$$\left(\frac{d^2 y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(-0.001) + \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right] = -1.703.$$

6.3 Maxima and Minima of a Tabulated Function

[B.C.A. (Kanpur) 2007, 2009]

Newton's forward interpolation formula yields:

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

Then differentiating it w.r.t. u, we have

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

But for maxima, or minima, $dy/du = 0$.

$$\therefore \quad \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 = 0 \quad \text{[retaining up to 3rd diff.]}$$

$$\Rightarrow \quad (\frac{1}{2}\Delta^3 y_0)\,u^2 + (\Delta^2 y_0 - \Delta^3 y_0)\,u + (\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0) = 0.$$

Then putting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table,

we can solve this quadratic in u. *The corresponding values of x* are given by $x = x_0 + uh$.

Example 6:* *From the table given below, for what values of x; y is minimum? Also find this value of y.

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

Solution: The difference table is given as under:

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		−0.016	
		0.019		0.000
5	0.259		−0.016	
		0.003		0.001
6	0.262		−0.015	
		−0.012		0.001
7	0.250		−0.014	
		−0.026		
8	0.224			

Assuming $x_0 = 3$, we get

$$y_0 = 0.205, \Delta y_0 = 0.035, \Delta^2 y_0 = -0.016 \text{ and } \Delta^3 y_0 = 0.$$

Then using Newton's forward difference formula, we get

$$y = 0.205 + u(0.035) + \frac{u(u-1)}{2}(-0.016) \qquad \text{...(1)}$$

Differentiating it w.r.t. u, we have

$$\frac{dy}{du} = 0.035 + \frac{2u-1}{2}(-0.016). \text{ For } y \text{ to be minimum, } dy/du = 0$$

$$\Rightarrow \quad 0.035 - 0.008(2u-1) = 0 \quad \Rightarrow u = 2.6875$$

and $x = x_0 + uh = 3 + 2.6875 \times 1 = 5.6875$.

Thus y is minimum at $x = 5.6875$.

$$\therefore \quad y_{\min} = 0.205 + 2.6875 \times 0.035 + \frac{1}{2}(2.6875 \times 1.6875)(-0.016) = 0.2628.$$

Example 7: ***Find the value of $f'(x)$ at $x = .04$ from the following table:***

x	.01	.02	.03	.04	.05	.06
$f(x)$	.1023	.1047	.1071	.1096	.1122	.1148

[B.C.A. (Kanpur) 2011]

Solution: Since we want to find the derivative at a point near the middle of the table, we may use any central difference formula. In this case we have used Bessel's formula. Here $x = .04$ and $h = .01$.

Consider a new variate $u = \dfrac{x - .04}{h}$

We want $f'(x)$ at $x = .04$. Again for $x = .04$, $u = 0$

and $\dfrac{d}{dx} f(u) = \dfrac{d\,du}{du\,dx} f(u) = \dfrac{d}{du} f(u) \dfrac{.1}{h} = \dfrac{f'(u)}{h}$

The difference table is as under:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
−3	.01	.1023				
			.0024.			
−2	.02	.1047		.0000		
			.0024		.0001	
−1	.03	.1071		.0001		−.0001
			.0025		0.000	
0	.04	.1096		.0001		−.0001
			.0026		−.0001	
1	.05	.1122		.0000		
			.0026			
2	.06	.1148				

Bessel's formula is

$$f(u) = \frac{1}{2}[f(0) + f(1)] + \left(u - \frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{2!}\frac{1}{2}[\Delta^2 f(-1) + \Delta^2 f(0)]$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\frac{1}{2}[\Delta^4 f(-1) + \Delta^4 f(-2)]$$

upto fourth differences.

Differentiating w.r.t. u and putting $u = 0$, we have

$$f'(0) = \Delta f(0) - \frac{1}{4}[\Delta^2 f(-1) + \Delta^2 f(0)] + \frac{1}{12}\Delta^3 f(-1) + \frac{1}{24}[\Delta^4 f(-1) + \Delta^4 f(-2)]$$

$$= .0026 - \frac{1}{4}[.0001 + .0000] + \frac{1}{12}[-.0001] + \frac{1}{24}[-.0001 - .0001]$$

$$= .0026 - \frac{.0001}{4} - \frac{.0001}{12} - \frac{.0002}{24} = \frac{.03075}{12}$$

$$\therefore \qquad f'(0) = \frac{f'(0)}{h} = \frac{.03075}{12 \times .01} = \frac{3.075}{12} = .2561.$$

Example 8: ***Find $f'(7.50)$ from the following table:***

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$y = f(x)$	.193	.195	.198	.201	.203	.206	.208

[B.C.A. (Rohilkhand) 2007, 2011; B.B.A. (Meerut) 2002, 2004]

Solution: Since we want to find the derivative near the middle of the table, we shall use one of the central difference formulae.

The Bessel's formula with a new variable u, given by $u = \dfrac{x - 7.50}{h}$ where $x_0 = 7.5$, $h = 0.01$ is

$$f(u) = \frac{1}{2}[f(0) + f(1)] + \left(u - \frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{2!}\,\frac{[\Delta^2 f(-1) + \Delta^2 f(0)]}{2}$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)\,u\,(u-1)(u-2)}{4!}\cdot\frac{[\Delta^4 f(-1) + \Delta^4 f(-2)]}{2}$$

$$+ \frac{\left(u - \frac{1}{2}\right)(u+1)\,u\,(u-1)(u-2)}{5!}\Delta^5 f(-2)$$

$$+ \frac{(u+2)(u+1)\,u\,(u-1)(u-2)(u-3)}{6!}\cdot\frac{[\Delta^6 f(-3) + \Delta^6 f(-2)]}{2} + \quad \ldots(1)$$

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
−3	7.47	.193						
			.002					
−2	7.48	.195		.001				
			.003		−.001			
−1	7.49	.198		.000		.000		
			.003		−.001		.003	
0	7.50	.201		−.001		.003		−.010
			.002		.002		−.007	
1	7.51	.203		.001		−.004		
			.003		−.002			
2	7.52	.206		−.001				
			.002					
3	7.53	.208						

Since $u = \dfrac{x - 7.50}{h}, \dfrac{du}{dx} = \dfrac{1}{h}, \quad \therefore \quad \dfrac{d}{dx}\{f(x)\} = \dfrac{d}{du} f(x) . \dfrac{du}{dx} = \dfrac{1}{h} f'(u)$

Now differentiating (1) w.r.t. u and putting $u = 0$, we get

$$f'(0) = .002 - \frac{1}{4}[-.001 + .001] + \frac{1}{12}(.002) + \frac{1}{24}[-.004 + .003] - \frac{1}{120}(-.007) - \frac{1}{120}(-.010)$$

$$= .002 + 0 + .0001666 - .0000416 + .0000583 + .0000833$$

$$= 0.0022666 \text{ (on simplification)}$$

$$\therefore \qquad f'(7.50) = \frac{1}{h} f'(0) = \frac{1}{.01}(.0022666) = .22666.$$

Example 9: ***Given the values of an emperical function $f(x)$ for certain values of x. Find (i) $f'(93)$, (ii) the value of x for which $f(x)$ is a maximum, (iii) the maximum value of $f(x)$ in the range of x.***

x	60	75	90	105	120
$f(x)$	28.2	38.2	43.2	40.9	37.7

Solution: (i) Since we require the derivative at a point near the middle of the table, so we shall use one of the central difference formulae. In this case, we have used Stirling's formula; The difference table is as under:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
−2	60	28.2				
			10.0			
−1	75	38.2		−5.0		
			5.0		−2.3	
0	90	43.2		−7.3		8.7
			−2.3		6.4	
1	105	40.9		−.9		
			−3.2			
2	120	37.7				

where a new variate $u = \dfrac{x - 90}{15}$, for $x = 93, u = 1/5 = 0.2$.

Now $\dfrac{d}{dx} f(u) = \dfrac{1}{h}\dfrac{d}{du} f(u) = \dfrac{1}{h} f'(u)$.

$$f(u) = f(0) + u\frac{\Delta f(0) + \Delta f(-1)}{2} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{u(u^2 - 1^2)}{3!} \times \frac{1}{2}[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{u^2(u^2 - 1^2)}{4!}\Delta^4 f(-2) + \ldots \qquad \ldots(1)$$

Differentiating w.r.t. u and putting $u = 0.2$; we get

$$f'(u) = \frac{\Delta f(0) + \Delta f(-1)}{2} + \frac{2\,(.2)}{2\,!}\Delta^2 f(-1)$$

$$+ \frac{3\,(.2)^2 - 1}{3\,!}\,\frac{1}{2}\,[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{4\,(.2)^3 - 2\,(.2)}{4\,!}\,\Delta^4 f(-2)$$

$$= \frac{-2.3 + (5.0)}{2} + .2\,(-7.3) + \frac{.12 - 1}{12}\,[6.4 - 2.3] + \frac{.032 - .4}{24}\,(8.7)$$

$$= \frac{2.7}{2} - 1.46 - \frac{.22\,(4.1)}{3} - \frac{.368\,(8.7)}{24}$$

$$= 1.35 - 1.46 - .30067 - .1334 = -.54407.$$

Now $$f'(x) = \frac{1}{h}\,f'(u) = \frac{1}{15}\,(-.54407) = -.036271.$$

(ii) In order to have $f(x)$ maximum, solve the equation

$$f'(x) = 0 \qquad \text{or} \quad f'(u) = 0$$

i.e. $$\frac{\Delta f(0) + \Delta f(-1)}{2} + \frac{2u}{2\,!}\,\Delta^2 f(-1) + \frac{3u^2 - 1}{3\,!}$$

$$\times \frac{1}{2}\,[\Delta^3 f(-1) + \Delta^3 f(-2)] + \frac{4u^3 - 2u}{4\,!}\,\Delta^4 f(-2) = 0$$

or $$1.35 + u\,(-7.3) + \frac{3u^2 - 1}{12}\,[4.1] + \frac{4u^3 - 2u}{24}\,(8.7) = 0$$

or $$16.20 - 87.6u + (3u^2 - 1)\,(4.1) + (2u^3 - u)\,8.7 = 0.$$

$$17.4u^3 + 12.3u^2 - 96.3u + 12.10 = 0$$

Solving this equation by Newton-Rephson method, we have

$$u = .128126. \quad \therefore \quad x = 90 + 15u = 90 + 15\,(0.128126) = 91.92189.$$

(iii) The value of $f(x)$ at this point is obtained by putting

$u = .128126$ in Stirling's formula (1) : $f(91.92189) = 43.2641.$

Example 10: ***Compute*** $f'''(15)$ ***given:***

x	2	4	9	13	16	21	29
$f(x)$	57	1345	66340	402052	1118209	4287844	21242820

[B.C.A. (Kashi) 2007]

Solution: In this case the values of the argument are unequally spaced, so we may use Newton's divided difference formula.

The divided difference table is as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
2	57					
		644				
4	1345		1765			
		12999		556		
9	66340		7881		45	
		83928		1186		1
13	402052		22113		64	
		238719		2274		1
16	1118209		49401		89	
		633927		4054		
21	4287844		114265			
		2119372				
29	21242820					

Since the fifth divided differences are constant, Newton's divided difference formula is

$$f(x) = f(a) + f(x-a) + \Delta f(a) + (x-a)(x-b)\,\Delta^2 f(a) + (x-a)(x-b)(x-c)\,\Delta^3 f(a)$$

$$+ (x-a)(x-b)(x-c)(x-d)\,\Delta^4 f(a) + (x-a)(x-b)(x-c)(x-d)(x-e)\,\Delta^5 f(a) + \ldots$$

Differentiating w.r.t. x thrice, we get

$$f'''(x) = 6\,\Delta^3 f(a) + [24x - 6(a+b+c+d)]\,\Delta^4 f(a)$$

$$[60x^2 - 24x(a+b+c+d) + 6(ab+cd) + (a+b)(c+d)$$

$$- e\,24x - 6(a+b+c+d)]\,\Delta^5 f(a).$$

Putting $a = 2, b = 4, c = 9, d = 13, e = 16, x = 5$ we get

$$f'''(5) = 6 \times 556 + [120 - 168]\,45 + [1500 - 3360 + 750 + 792 - 16(120 - 168)]$$

$$= 3336 - 2160 + 450 = 1626.$$

Example 11: ***Find the first three derivatives of the function tabulated below at the point*** $x = 2.5$:

x	1.5	1.9	2.5	3.2	4.3	5.9
$f(x)$	3.375	6.059	13.625	29.368	73.907	196.579

Solution: In this case the values of the argument are unequally spaced, we may use here Newton's divided difference formula.

The divided difference table is as under:

x	$f(x)$	Δ	Δ^2	Δ^3
1.5	3.375			
		6.71		
1.9	6.059		5.90	
		12.61		1
2.5	13.625		7.60	
		22.49		1
3.2	29.368		10.00	
		40.49		1
4.3	73.907		13.40	
		76.67		
5.9	196.579			

We observe that the third divided differences are constant.

Again Newton's divided difference formula is given by

$$f(x) = f(a) + (x-a)\,\Delta f(a) + (x-a)(x-b)\,\Delta^2 f(a) + (x-a)(x-b)(x-c)\,\Delta^3 f(a)$$

taking upto third differences.

On differentiating w.r.t. x thrice, we get

$$f'(x) = \Delta f(a) + [2x - a - b]\,\Delta^2 f(a) + [3x^2 - 2x(a+b+c) + ab + ac + bc]\,\Delta^3 f(a)$$

$$f''(x) = 2\,\Delta^2 f(a) + [6x - 2(a+b+c)]\,\Delta^3 f(a)$$

$$f'''(x) = 6\,\Delta^3 f(a)$$

Here $a = 1.5, b = 1.9, c = 2.5$ On putting $x = 2.5$, we get

$$f'(2.5) = 6.710 + [5 - 3.4]\,5.9 + [3 \times 6.25 - 5(1.5 + 1.9 + 2.5) + 11.35]\,.1$$

$$= 6.710 + 9.44 + .600 = 16.750$$

$$f''(2.5) = 2[5.9] + [15.00 - 2 \times 5.9]\,1 = 15.00,$$

$$f'''(2.5) = 6 \times 1 = 6.00.$$

Example 12: ***Using divided difference, find the value of $f(8)$, given that $f(6)=1.556$, $f(7)=1.690$, $f(9)=1.908$, $f(12)=2.158$.***

[B.C.A. (Bhopal) 2007, 2012; B.C.A. (I.P.U.) Delhi 2007, 2012]

Solution: Divided difference formula is given by

$$f(x) = f(x_0) + (x-x_0)\,\Delta f(x_0) + (x-x_0)(x-x_1)\,\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\,\Delta^4 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\,\Delta^5 f(x_0) \ldots \qquad \ldots(1)$$

The divided difference table is given by:

x	$f(x)$	Δ	Δ^2	Δ^3
6	1.556			
		.134		
7	1.690		$-.0083$	
		.109		.00051
9	1.908		$-.0052$	
		.083		
12	2.158			

Differentiating (1) w.r.t. to x; we get

$$f'(x) = \Delta f(x_0) + [2x - x_0 - x_1]\,\Delta^2 f(x_0)$$

$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_0x_2]\,\Delta^3 f(x_0)$$

Putting $x_0 = 6, x_1 = 7, x_2 = 9, x_3 = 12, x = 8$ and the values of the divided differences from the table, we get

$$f'(8) = .134 + [2\times 8 - 6 - 7](-.0083)$$

$$+ [3\times 64 - 2\times 8(6+7+9) + 6\times 7 + 7\times 9 + 6\times 9](.00051)$$

$$= .134 - .0249 + (192 - 352 + 159)(.00051)$$

$$= .134 - .0249 - .00051 = .134 - .0254 = .10859$$

Example 13: ***Given the following pairs of values of x and $y = f(x)$:***

x	1	2	4	8	10
$y = f(x)$	0	1	5	21	27

Determine numerically the first derivative at $x = 4$.

Solution: In this case the value of the argument are not equally spaced, so we shall use Newton's divided difference formula. The divided difference formula table is:

x	y	Δ	Δ^2	Δ^3	Δ^4
1	0				
		1			
2	1		$\frac{1}{3}$		
		2		0	
4	5		$\frac{1}{3}$		$-\frac{1}{144}$
		4		$-\frac{1}{16}$	
8	21		$-\frac{1}{6}$		
		3			
10	27				

Newton's divided difference formula is

$$f(x) = f(a) + (x-a)\,\Delta f(a) + (x-a)(x-b)\,\Delta^2 f(a) + (x-a)(x-b)(x-c)\,\Delta^3 f(a)$$
$$+ (x-a)(x-b)(x-c)(x-d)\,\Delta^4 f(a) + \ldots$$

Differentiating this w.r.t. x and putting $x = c$, we have

$$f'(c) = \Delta f(a) + (2c-a-b)\,\Delta^2 f(a) + (c-a)(c-b)\,\Delta^3 f(a) + (c-a)(c-b)(c-d)\,\Delta^4 f(a).$$

Putting $a = 1, b = 2, c = 4, d = 8, e = 10$ and the values of differences from the table, we get

$$f'(4) = 1 + 5 \times \frac{1}{3} + 3 \times 2 \times 0 + 3 \times 0 + 3 \times 2 \times (-4) \times \left(-\frac{1}{144}\right)$$
$$= 1 + 1.666 + .1666 = 2.832.$$

Example 14: ***Find*** $f'(5)$ ***from the following table:***

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

[B.C.A. (Meerut) 2004]

Solution: The value of the argument are not equally spaced, so we shall use Newton's divided difference formula. The divided difference table is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	4			
		11		
2	26		7	
		32		1
3	58		11	
		54		1
4	112		16	
		118		1
7	466		22	
		228		
9	922			

Newton's divided difference formula, if third differences are constant is given by

$$f(x) = f(x_0) + (x-x_0)\,f(x_0, x_1) + (x-x_0)(x-x_1)\,f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\,f(x_0, x_1, x_2, x_3).$$

Now differentiating this w.r.t. x, we get

$$f'(x) = f(x_0, x_1) + \{(x-x_1) + (x-x_0)\}\,f(x_0, x_1, x_2)$$
$$+ \{(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)\}\,f(x_0, x_1, x_2, x_3).$$

Hence putting $x = 5, x_0 = 0, x_1 = 2, x_2 = 3, x_3 = 4$ and the values of various divided differences, we get

$$f'(5) = 11 + \{(5-2)(5-0)\} \times 7 + \{(5-2)(5-3) + (5-0)(5-3) + (5-0)(5-2)\} \times 1$$
$$= 11 + 56 + 31 = 98$$

Example 15: ***Assuming Stirling's formula, obtain the following approximation:***

$$\boldsymbol{\frac{d}{dx}f(x)=\frac{2}{3}[f(x+1)-f(x-1)]-\frac{1}{12}[f(x+2)-f(x-2)]}$$ ***upto third differences.***

Solution: The Stirling's formula is

$$f(x)=f(0)+x\,\frac{\Delta f(0)+\Delta f(-1)}{2}+\frac{x^2}{2\,!}\Delta^2 f(-1)+\frac{x\,(x^2-1)}{3\,!}\cdot\frac{\Delta^3 f(-1)+\Delta^3 f(-2)}{2}+\ldots$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}f(x)=\frac{\Delta f(0)+\Delta f(-1)}{2}+\frac{2x}{2\,!}\Delta^2 f(-1)+\frac{(3x^2-1)}{3\,!}\cdot\frac{\Delta^3 f(-1)+\Delta^3 f(-2)}{2}$$

(taking upto third differences only)

Putting $x=0$, we get

$$\frac{d}{dx}f(0)=\frac{\Delta f(0)+\Delta f(-1)}{2}-\frac{1}{12}[\Delta^3 f(-1)+\Delta^3 f(-2)]$$

Now putting $\Delta f(0)=f(1)-f(0),\Delta f(-1)=f(0)-f(-1)$

$$\Delta^3 f(-1)=f(1)-3f(1)+3f(0)-f(-1),\ \Delta^3 f(-2)=f(1)-3f(0)+3f(-1)-f(-2)$$

and then simplifying, we get

$$\frac{d}{dx}f(0)=\frac{2}{3}[f(1)-f(-1)]-\frac{1}{12}[f(2)-f(-2)]$$

Shifting the origin to x, we have

$$\frac{d}{dx}f(x)=\frac{2}{3}[f(x+1)-f(x-1)]-\frac{1}{12}[f(x+2)-f(x-2)].$$

Example 16: ***Assuming Bessel's Interpolation formula, prove that***

$$\boldsymbol{\frac{d}{dx}(y_x)=\Delta y_{x-1/2}-\frac{1}{24}\Delta^3 y_{x-3/2}+\ldots}$$

[B.C.A. (Agra) 2006; B.C.A. (Kanpur) 2003, 2005; B.C.A. (Lucknow) 2003, 2007]

Solution: The Bessel's formula is

$$y_x=\frac{y_0+y_1}{2}+\left(x-\frac{1}{2}\right)\Delta y_0+\frac{x\,(x-1)}{2\,!}\cdot\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\frac{\left(x-\frac{1}{2}\right)x\,(x-1)}{3\,!}\Delta^3 y_{-1}+\ldots \quad \ldots(1)$$

Changing x to $x+\frac{1}{2}$, we get

$$y_{x+1/2}=\frac{y_0+y_1}{2}+x\,\Delta y_0+\frac{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{2\,!}\cdot\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}$$

$$+\frac{x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{3}\Delta^3 y_{-1}+\ldots \quad \ldots(2)$$

Differentiating (2) w.r.t. x, we get

$$\frac{d}{dx}(y_{x+1/2}) = \Delta y_0 + \frac{2x}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(x^2 - \frac{1}{4}\right)}{3!}\Delta^3 y_{-1} + \ldots$$

Putting $x = 0$, we get

$$\frac{d}{dx}(y_{x+1/2}) = \Delta y_0 - \frac{1}{24}\Delta^3 y_{-1} + \ldots$$

Now shifting the origin from $x = 0$ to $x - \frac{1}{2}$, we get

$$\frac{d}{dx}(y_x) = \Delta y_{x-1/2} - \frac{1}{24}\Delta^3 y_{x-3/2} + \ldots$$

Example 17: ***Prove that***

$$y' = \frac{1}{h}\left(\delta y - \frac{1}{24}\delta^3 y + \frac{3}{640}\delta^5 y + \ldots\right)$$

and

$$y'' = \frac{1}{h^2}\left(\delta^2 y - \frac{1}{12}\delta^4 y + \frac{1}{90}\delta^6 y - \ldots\right)$$

Solution: We know that

$$\delta = E^{1/2} - E^{-1/2} = e^{hD/2} - e^{-hD/2} \qquad (\because \quad E \equiv e^{hD})$$

$$= 2 \sin h \frac{hD}{2}$$

or

$$\frac{hD}{2} = \sin h^{-1}\frac{\delta}{2} \qquad \ldots(1)$$

by Taylor's expansion, we have from (1)

$$\sin h^{-1} x = x - \frac{x^3}{6} + \frac{3}{40}x^5 - \ldots$$

Therefore by this expression, we have

$$\frac{hD}{2} = \frac{\delta}{2} - \frac{\delta^3}{6 \times 8} + \frac{3}{40 \times 32}\delta^5 - \ldots$$

or

$$D = \frac{1}{h}\left[\delta - \frac{1}{24}\delta^3 + \frac{3}{640}\delta^5 - \ldots\right] \qquad \ldots(2)$$

Again operating these operators on y, we get

$$D\,y = y' = \frac{1}{h}\left[\delta y - \frac{1}{24}\delta^3 y + \frac{3}{640}\delta^5 y - \ldots\right] \qquad \ldots(3)$$

From (2), we have

$$D^2 = \frac{1}{h^2}\left[\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \ldots\right]$$

Operating these operators on y, we get

$$D^2 y = y'' = \frac{1}{h^2}\left[\delta^2 y - \frac{1}{12}\delta^4 y + \frac{1}{90}\delta^6 y - \ldots\right]$$

Example 18: ***Show that*** $y' = \frac{\mu}{h}\left(\delta y - \frac{1}{6}\delta^3 y + \frac{1}{30}\delta^5 y - \ldots\right)$

and $$y'_0 = \frac{1}{2h}\left[(y_1 - y_{-1}) - \frac{1}{6}(\delta^2 y_1 - \delta^2 y_{-1}) + \frac{1}{30}(\delta^4 y_1 - \delta^4 y_{-1})\ldots\right]$$

Solution: We know that $\mu^2 = 1 + \frac{\delta^2}{4}$.

$$\Rightarrow \qquad \mu^{-1} = \left(1 + \frac{\delta^2}{4}\right)^{-1/2} = 1 - \frac{\delta^2}{8} + \frac{3}{128}\delta^4 - \ldots$$

Also from equation (2) of Ex. 17, we have

$$\frac{hD}{\delta} = 1 - \frac{\delta^2}{24} + \frac{3}{640}\delta^4 - \ldots$$

Now $$\frac{hD}{\delta\mu} = \frac{hD}{\delta}\mu^{-1} = \left[1 - \frac{\delta^2}{24} + \frac{3}{640}\delta^4 - \ldots\right] \times \left[1 - \frac{\delta^2}{8} + \frac{3}{128}\delta^4 - \ldots\right]$$

$$= 1 - \frac{\delta^2}{6} + \frac{1}{30}\delta^4 - \ldots$$

$$\Rightarrow \qquad D = \frac{\delta\mu}{h}\left[1 - \frac{\delta^2}{6} + \frac{1}{30}\delta^4 - \ldots\right] = \frac{\mu}{h}\left[\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \ldots\right]$$

Again, operating these operators on y, we obtain

$$Dy = \frac{\mu}{h}\left[\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \ldots\right]y \quad \text{or} \quad y' = \frac{\mu}{h}\left[\delta y - \frac{\delta^3 y}{6} + \frac{\delta^5 y}{30} - \ldots\right] \qquad \ldots(1)$$

But by definition, we have

$$\delta = E^{1/2} - E^{-1/2} \text{ and } \mu = \frac{1}{2}(E^{1/2} + E^{-1/2}).$$

$$\Rightarrow \qquad \delta\mu = \frac{1}{2}(E - E^{-1}) = \mu\delta$$

$$\Rightarrow \qquad (\delta\mu)\, y_0 = \frac{1}{2}(E - E^{-1})\, y_0 = \frac{1}{2}(y_1 - y_{-1}) = \mu\delta\, y_0 \qquad \ldots(2)$$

Hence from (1), we get

$$y'_0 = \frac{1}{h}\left[\mu\,\delta y_0 - \frac{\mu\,\delta^3}{6}y_0 + \mu\frac{\delta^5}{30}y_0 - \ldots\right]$$

$$= \frac{1}{2h}\left[(y_1 - y_{-1}) - \frac{1}{6}(\delta^2 y_1 - \delta^2 y_{-1}) + \frac{1}{30}(\delta^4 y_1 - \delta^4 y_{-1}) - \ldots\right]$$

Example 19: ***Prove that*** $y' = \frac{1}{h}\left(\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \ldots\right)$

and $$y'' = \frac{1}{h^2}\left(\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \ldots\right).$$

Solution: We have $E \equiv e^{hD}$.

$$\Rightarrow \qquad 1+\Delta \equiv e^{hD}.$$

$$\Rightarrow \qquad hD \equiv \log(1+\Delta)$$

$$\Rightarrow \qquad D \equiv \frac{1}{h}\log(1+\Delta)$$

$$\Rightarrow \qquad Dy \equiv \frac{1}{h}\log(1+\Delta)\ y$$

$$\equiv \frac{1}{h}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^3}{4} + \ldots\right\} y.$$

$$\Rightarrow \qquad y' \equiv \frac{1}{h}\left\{\Delta y - \frac{\Delta^2 y}{2} + \frac{\Delta^3 y}{3} - \frac{\Delta^4 y}{4} + \ldots\right\}$$

Again $\quad e^{-hD} \equiv 1-\nabla. \qquad \Rightarrow \quad D \equiv -\frac{1}{h}\log(1-\nabla).$

Now, $\quad y'' = D^2 y = \left\{-\frac{1}{h}\log(1-\nabla)\right\}^2 y = \frac{1}{h^2}\{\log(1-\nabla)\}^2 y$

$$= \frac{1}{h^2}\{\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\}^2 y = \frac{1}{h^2}\left\{\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right\} y$$

$$= \frac{1}{h^2}\left\{\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \ldots\right\}$$

Example 20:* *Prove that

$$\frac{d}{dx}(y_x) = \frac{1}{h}(y_{x+h} - y_{x-h}) - \frac{1}{2h}(y_{x+2h} - y_{x-2h}) + \frac{1}{3h}(y_{x+3h} - y_{x-3h})$$

[B.C.A. (Rohtak) 2006, 2012]

Solution: **R.H.S.** $= \frac{1}{h}\{y_{x+h} - \frac{1}{2}y_{x+2h} + \frac{1}{3}y_{x+3h} - \ldots\}$

$$-\frac{1}{h}\left\{y_{x-h} - \frac{1}{2}y_{x-2h} + \frac{1}{3}y_{x-3h} - \ldots\right\}$$

$$= \frac{1}{h}[\{Ey_x - \frac{1}{2}E^2 y_x + \frac{1}{3}E^3 y_x - \ldots\} - \{E^{-1}y_x - \frac{1}{2}E^{-2}y_x + \frac{1}{3}E^{-3}y_x - \ldots\}]$$

$$= \frac{1}{h}[\{\log(1+E)\}\ y_x - \{\log(1+E^{-1})\}\ y_x]..$$

$$= \frac{1}{h}\log\left(\frac{1+E}{1+E^{-1}}\right) y_x = \left\{\frac{1}{h}\log E\right\} y_x = \left(\frac{1}{h}\log e^{hD}\right) y_x \left(\because \quad E \equiv e^{hD}\right)$$

$= Dy_x = \frac{d}{dx}(y_x) =$ **L.H.S.**

Problem Set

1. Find $\frac{dy}{dx}$ at $x = 1$ from the following table:

x	.7	.8	.9	1.0	1.1	1.2	1.3
y	0.644218	0.717356	0.783327	0.841471	0.891207	0.932039	0.963558

[B.C.A. (Meerut) 2003]

2. Find the first two derivatives of $f(x)$ at $x = 1$ from the following table:

x	−2	−1	0	1	2	3	4
$f(x)$	104	17	0	−1	8	69	272

3. Find $f'(6)$ from the following table:

x	0	1	3	4	5	7	9
$f(x)$	150	108	0	−54	−100	−144	−84

[B.B.A. (Meerut) 2005]

4. Show that the expressions given below are approximations to the third derivatives of y_x:

(i) $\Delta^3 y_0 + \left(x - \frac{3}{2}\right) \Delta^4 y_0,$ **[B.C.A. (Lucknow) 2002]**

(ii) $\Delta^3 y_{-1} + \left(x - \frac{1}{2}\right) . \frac{1}{2} (\Delta^4 y_{-2} + \Delta^4 y_{-1}).$

5. Using divided differences, find the value of $f'(8)$, given that $f(6) = 1.556$, $f(7) = 1.690$, $f(9) = 1.908$, $f(12) = 2.158$.

6. Use the following data to find $f(1)$

x	3	5	11	27	34
$f(x)$	−13	23	899	17315	35606

7. Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ of $y = x^{1/3}$ at $x = 50$, from the following table:

x	$y = x^{1/3}$
50	3.6840
51	3.7084
52	3.7325
53	3.7563
54	3.7798
55	3.8030
56	3.8259

8. For what value of x is the following tabulated function minimum:

x	0.2	0.3	0.4	0.5	0.6	0.7
y	.9182	.8975	.8873	.8862	.8935	.9086

9. Assuming Bessel's interpolation formula, deduce the following approximations:

$$\frac{d}{dx} y_x = \Delta y_{x-1/2} - \frac{1}{24} \Delta^3 y_{x-3/2}$$

$$\frac{d^2}{d x^2} (y_x) = \frac{1}{2} [\Delta^2 y_{x-3/2} + \Delta^2 y_{x-1/3}] + \dots$$

$$\frac{d^3}{d x^3} (y_x) = \Delta^2 y_{x-3/2} + \dots$$

$$\frac{d^4}{d x^4} (y_x) = \Delta^3 y_{-1} + \left(x - \frac{1}{2}\right) . \frac{1}{2} (\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \dots$$

10. By using Stirling formula of degree six, shown that

(i) $y'(x_0) = \dfrac{1}{h}\left(\delta\mu\, y_0 - \dfrac{1}{6} \delta^3 \mu\, y_0 + \dfrac{1}{30} \delta^5 \mu\, y_0 + \dots\right)$

(ii) $y''(x_0) = \dfrac{1}{h}\left[\delta^2 f_0 - \dfrac{1}{12} \delta^4 f_0 + \dots\right]$

11. Prove the following formula

$$y' = \frac{1}{h}\left(\Delta y - \frac{1}{2} \Delta^2 y + \frac{1}{3} \Delta^3 y - \frac{1}{4} \Delta^4 y + \dots\right)$$

and

$$y'' = \frac{1}{h}\left(\nabla^2 y + \nabla^3 y + \frac{11}{12} \nabla^4 y + \dots\right)$$

12. Use Stirling formula to find the first derivative of the function $y = 2\,e^x - x - 1$ tabulated below at the point $x = 0.6$

x	y
0.4	1.5836494
0.5	1.7974426
0.6	2.0442376
0.7	2.3275054
0.8	2.6510818

Compare with the true value which is 2.044238.

13. Find $f'(1)$ for $f(x) = \dfrac{1}{(1 + x^2)}$ using the following data:

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	.2500	.2268	.2066	.1890	.1736

14. The function $y = \sin x$ is tabulated in the scheme below. Find the derivative at the point $x = 1$.

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y	.644218	.717356	.783327	.841471	.891207	.932039	.963558

15. A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values for the time t seconds. Find the velocity and acceleration of the slider, when $t = 0.3$ second

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Answers

1.	0.54030
2.	$1, 6$
3.	-23
5.	0.109
6.	233
7.	$.02455, -.0003$
8.	0.4623
12.	2.644225
13.	-0.5031
14.	0.54030
15.	Velocity $= 5.34$ cm/sec. Acceleration $= -45.6$ cm/sec^2.

❍❍❍

Unit-3

Chapter 7

Numerical Integration

7.1 Introduction

It is the process of finding or evaluating a definite integral $I = \int_a^b f(x)\,dx$ from a set of numerical values of the integrand $f(x)$. If it is applied to the integration of function of a single variable the process is known as ***quadrature.*** The problem of numerical integration is solved by first approximating the integrand by a polynomial with the help of an interpolation formula and then integrating this expression between the desired limits. Thus to evaluate the definite integral $\int_a^b f(x)\,dx$, first express the function $f(x)$ by an interpolation formula say $p(x)$ and then integrate $f(x)$ between the limits a and b

i.e.
$$\int_a^b f(x)\,dx \sim \int_a^b p(x)\,dx$$

The error in E in such type of approximation is given by

$$\int_a^b f(x)\,dx - \int_a^b p(x)\,dx = \int_a^b [f(x) - p(x)]\,dx\,.$$

7.2 A General Quadrature Formula for Equidistant Ordinates

[Meerut 2000, 2002, 2003, 2005; Avadh 2003, 2004, 2005; Bhopal 2002; Agra 2003]

Let $I = \int_a^b y\,dx$ where $y = f(x)$. Let $f(x)$ be given for certain equally distant values of arguments, say $x_0, x_0 + h, x_0 + 2h, \ldots\ldots$ Let the range $(b - a)$ be divided into n equal parts, each of which is of width h, *i.e.* $b - a = nh.$

Let $x_0 = a, x_1 = x_0 + h = a + h, x_2 = a + 2h, \ldots, x_n = a + nh = b,$

we have assumed that the $(n+1)$ ordinates $y_0, y_1, \ldots, y_n$ are at equal intervals.

$$\therefore \quad I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y_x\,dx = \int_0^n y_{x_0+uh}\,h\,du.$$

where new variate $u = \dfrac{x - x_0}{h}; \quad dx = h\,du$

or

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0\right]du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots \text{ up to } (n+1) \text{ terms}\right]. \quad \ldots(1)$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \ldots\right] \quad \ldots(2)$$

This is known as ***Newton-Cote's quadrature*** formula and is a general formula. We deduce the following important quadrature rules by taking $n = 1, 2, 3, \ldots$

7.3 The Trapezoidal Rule

Put $n = 1$ in (1) and neglect second and higher differences when $n = 1$, the interval of integration will be from x_0 to $x_0 + h$, and there are only two functional values y_0 and y_1 in this interval, with only two values, there can be no differences higher than the first. Thus we have

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{y_1 - y_0}{2}\right] = h\left[\frac{y_0 + y_1}{2}\right]$$

Similarly $\displaystyle\int_{x_0+h}^{x_0+2h} y\,dx = h\left(\frac{y_1 + y_2}{2}\right)$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_{x_0+\overline{(n-1)}h}^{x_0+nh} y\,dx = h\left(\frac{y_{n-1} + y_n}{2}\right)$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = h\,.\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \ldots + y_{n-1})\right]$$

= distance between two consecutive ordinates

× [mean on the first and the last ordinates + sum of all the intermediate ordinates]

This rule is called the ***Trapezoidal Rule.*** ...(1)

Note: Here we have taken y, a polynomial of the first degree in x or a straight line.

7.4 Simpson's One-Third Rule

[B.C.A. (Bhopal) 2002; B.C.A. (Avadh) 2005, 2007, 2010]

Put $n = 2$ in (1) of **7.2** and neglect third and higher differences. [In this case interval of integration will be from x_0 to $x_0 + 2h$ and there will be only three functional values y_0, y_1 and y_2: this with these values, there will be no differences higher than the second].

$$\therefore \qquad \int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + 2(y_1 - y_0) + \frac{\left(\frac{8}{3} - 2\right)}{2}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $\displaystyle\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

… … … … … … … … … … …

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \qquad (n \text{ is even});$$

on adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

= (One third of the distance between two consecutive ordinates) × [(Sum of the extreme ordinates) + 4 (Sum of the odd ordinates) + 2 (Sum of even ordinates)] ...(1)

This formula is known as ***Simpson's one-third rule.*** It is also called as ***parabolic formula.***

Note: In this case, we have neglected all differences above the second; so y will be polynomial of second degree only

i.e. $y = ax^2 + bx + c$

7.5 Simpson's Three-Eighth's Rule

[Meerut 2007; Bhopal 2002; Avadh 2005]

Put $n = 3$ in (1) of **7.2** and neglect all differences above the third. In this case only four functional values will exist and consequently there will be no differences higher than the third. Thus we get

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0)\right.$$

$$\left. + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3].$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}[(y_3 + 3y_4 + 3y_5 + y_6)]$$

… … … … … …

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n],$$

Where n is multiple of 3.

Adding all these integrals, we have, when n is a multiple of 3,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1})$$

$$+ 2(y_3 + y_6 + \ldots + y_{n-3})]. \qquad \ldots(1)$$

This formula is known as ***Simpson's three-eighth's rule.*** *Applying this rule, the number of sub-intervals should be taken as multiple of three.*

Note: Since we have neglected all differences above the third so y will be polynomial of the third degree.

i.e. $y = ax^3 + bx^2 + cx + d$

SOLVED EXAMPLES

Example 1: ***Compute the value of the definite integral***

$$\int_{.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$$

by (i) the Trapezoidal rule, (ii) Simpson's one-third rule, (iii) Simpson's three-eighth rule, (iv) Weddle's rule. After finding the true value of the integral, compare the errors in the four cases.

Solution: In this case divide the range of integration into twelve equal parts by taking $h = .1$ and then the values of the function $y = \sin x - \log_e x + e^x$ are computed for each point of sub-division which are as under:

	x	**sin x**	**$\log_e x = \log_{10} x$ $\times 2.30258$**	e^x	**$y = \sin x$ $- \log_e x + e^x$**
x_0	0.2	.19867	−1.60943	1.22140	3.02950
$x_0 + h$	0.3	.29552	−1.20397	1.34986	2.84935
$x_0 + 2h$	0.4	.38942	−.91629	1.49182	2.79753
$x_0 + 3h$	0.5	.47943	−.69315	1.64872	2.82130
$x_0 + 4h$	0.6	.56464	−.51083	1.82212	2.89759
$x_0 + 5h$	0.7	.64422	−.35667	2.01375	3.01464
$x_0 + 6h$	0.8	.71736	−.22314	2.22554	3.16604
$x_0 + 7h$	0.9	.78333	−.10537	2.45960	3.34830
$x_0 + 8h$	1.0	.84147	.00000	2.71828	3.55975
$x_0 + 9h$	1.1	.89121	.09530	3.00417	3.80008
$x_0 + 10h$	1.2	.93204	.18232	3.32012	4.06984
$x_0 + 11h$	1.3	.96356	.26236	3.66930	4.37050
$x_0 + 12h$	1.4	.98545	.33647	4.05520	4.70418

(i) By Trapezoidal Rule, we have

$$\int_{.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = \int_{.2}^{.2 + 12 \times .1} y\, dx$$

Here $n = 12, h = .1, x_0 = .2, y = \sin x - \log_e x + e^x$.

Again we know that

$$\int_{x_0}^{x_0 + nh} = \frac{h}{2} [y_0 + y_n + 2 (y_1 + y_2 + \ldots + y_{n-1})].$$

Substituting the values of $h, y_0, y_n, \ldots$, in the above formula, we get

$$\int_{.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = \frac{.1}{2} [3.02950 + 4.70418 + 2 (2.8493 + \ldots + 4.37050)]$$

$$= \frac{1}{20} [7.73368 + 2 (36.69492)] = 4.05617.$$

(ii) By Simpson's '$\frac{1}{3}$' Rule, we have

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)].$$

$$= \frac{.1}{3}(7.73368 + 4(20.20415) + 3(16.49075)]$$

$$= \frac{.1}{3}[121.53186] = 4.05106.$$

(iii) By Simpson's '$\frac{3}{8}$' Rule, we have

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \ldots) + 2(y_3 + y_6 + \ldots)]$$

$$= \frac{3(.1)}{8}[7.73368 + (26.90750) + 2(9.78742)] = 4.05116.$$

(iv) By Weddle's Rule, we have

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}[y_0 + y_n + 5(y_1 + y_5 + y_7 + y_{11} + \ldots)$$

$$+ y_2 + y_4 + y_8 + y_{10} + 2y_6 + 6(y_3 + y_9)]$$

$$= \frac{3(.1)}{10}[7.73360 + 5(13.58879) + 13.32471$$

$$+ 2(3.16604) + 6(6.62138)]$$

$$= 4.05098.$$

Again the value of the integral is

$$I = \int_{.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$$

$$= \left[-\cos x - x(\log_e x - 1) + e^x\right]_{.2}^{1.4} = 4.05095.$$

Therefore the errors are:

due to Trapezoidal Rule	$-.00522$
due to Simpson's '$\frac{1}{3}$' Rule	$-.00011$
due to Simpson's '$\frac{3}{8}$' Rule	$-.00021$
due to Weddle's Rule	$-.00003$

We observe that Weddle's rule is more accurate than other rules.

Example 2: ***A curve is drawn to pass through the points given by the following table:***

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, the x-axis and the lines $x = 1$, $x = 4$.

[B.C.A. (Meerut) 2002, 2004; B.C.A. (I.P.U. Delhi) 2007, 2012]

Solution: By Simpson's (1/3) Rule, we have required area

$$= \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n]$$

$$= \frac{0.5}{3}[2 + 9.6 + 5.4 + 11.2 + 10.4 + 2.1]$$

$$= \frac{5}{30} \times 40.7 = \frac{20.35}{3} = 6.783 \text{ units of area}$$

By Weddle's Rule, we have required area

$$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3 \times 0.5}{10}[2 + 12.0 + 2.7 + 16.8 + 3 + 13.0 + 2.1] = 7.74 \text{ units of area}$$

Example 3: ***Calculate by Simpson's rule an approximate value of $\int_{-3}^{3} x^4\,dx$ by taking seven equidistant ordinates. Compare it with the exact value and the value obtained by using the Trapezoidal rule.*** **[B.C.A. (Meerut) 2002, 2005]**

Solution: Let us divide the range of integration $(-3, 3)$ into six equal parts each of width $= \frac{3-(-3)}{6} = 1$ unit. Hence $h = 1$, value of the function for each point of sub- division are as given below:

x	$y = x^4$
$x_0 = -3$	$(-3)^4 = 81$
$x_0 + h = -2$	$(-2)^4 = 16$
$x_0 + 2h = -1$	$(-1)^4 = 1$
$x_0 + 3h = 0$	$0^4 = 0$
$x_0 + 4h = 1$	$1^4 = 1$
$x_0 + 5h = 2$	$2^4 = 16$
$x_0 + 6h = 3$	$3^4 = 81$

Now, by Simpson's rule, we obtain

$$\int_{-3}^{3} x^4\,dx = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[162 + 4 \times 32 + 2 \times 2] = \frac{1}{3} \times 294 = 98.$$

But the exact value of

$$\int_{-3}^{3} x^4\,dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{1}{5}\left[(3)^5 - (-3)^5\right] = \frac{1}{5} \times 486 = 97.2$$

Also by Trapezoidal rule, we have

$$\int_{-3}^{3} x^4\, dx = \frac{h}{2}\left[y_0 + y_6 + 2\,(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[162 + 2 \times 34\right] = 115\,.$$

Thus it is obvious that the Trapezoidal rule does not give an accurate result. In general Simpson's rule give a better result than Trapezoidal rule.

Example 4: ***Evaluate*** $\int_0^1 \frac{dx}{1 + x^2}$ ***by using Simpson's '$\frac{1}{3}$' and '$\frac{3}{8}$' rule. Hence obtain the approximate value of*** π ***in each ease.***

[Delhi (H) 2010; Meerut 2001, 2003; I.A.S. 2000, 2006; Avadh 2004, 2010; B.C.A. (Agra) 2005, 2011; B.C.A. (Lucknow) 2012; B.C.A. (Kanpur) 2007]

Solution: Divide the range into six equal parts each of with $h = \frac{1}{6}$ and compute the value of $y = \frac{1}{1 + x^2}$ at each point of sub-division.

These values are given below:

x		$y_x = 1/(1 + x^2)$
0	x_0	$(1/1) = 1.0000$
1/6	$x_0 + h$	$36/37 = 0.97297$
2/6	$x_0 + 2h$	$36/40 = 0.90000$
3/6	$x_0 + 3h$	$36/45 = .80000$
4/6	$x_0 + 4h$	$36/52 = .69231$
5/6	$x_0 + 5h$	$36/61 = .59016$
1	$x_0 + 6h$	$1/2 = .50000$

By Simpson's '$\frac{1}{3}$' rule, we have

$$\int_{x_0}^{x_0 + 6h} y\, dx = \frac{h}{3}\,[y_0 + y_6 + 4\,(y_1 + y_3 + y_5) + 2\,(y_2 + y_4)]$$

$$= \frac{1}{18}\,[1.00 + .500 + 4\,(.97297 + .80000 + .59016)]$$

$$+ 2\,(.900 + .69231)] \qquad ...(1)$$

$$= .785395 \text{ (on simplification).}$$

By Simpson's '$\frac{3}{8}$' rule, we have

$$\int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{1}{16}[1.5000 + 3(.97297 + .900 + .69231 + .59016) + 2 \times (.800)]$$

$$= .785395. \qquad ...(2)$$

Again, $$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4} \qquad ...(3)$$

From (1) and (2), we get $\pi = 3.14188$ and from (2) and (3), we get $\pi = 3.141580$.

Example 5:* *Evaluate $\int_0^6 \frac{dx}{1+x^2}$ ***by using (i) Trapezoidal rule***

(ii) Simpson's '$\frac{1}{3}$' rule (iii) Simpson's '$\frac{3}{8}$' rule, and (iv) Weddle's rule

[B.C.A. (Kurukshetra) 2004, 2007; B.C.A. (Rohilkhand) 2008, 2010]

Solution: We divide the range (0, 6) into six equal parts each of width $h = 1$ and compute the values of $y = \frac{1}{1+x^2}$ at each point of sub-division.

The computed values are as follows:

	x_0	x_0+h	x_0+2h	x_0+3h	x_0+4h	x_0+5h	x_0+6h
x :	0	1	2	3	4	5	6
y_x :	1.00	.500	.200	.100	.058824	.038462	.027027

(i) Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(1 + .027) + 2(.5 + .2 + .1 + .0588 + .0385)] = 1.4108.$$

(ii) Simpson's (1/3) rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(iii) Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.$$

(iv) Weddle's rule,

$$\int_0^6 \frac{dxc}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027]$$

$$= 1.3735$$

Also $\int_0^6 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^6 = \tan^{-1} 6 = 1.4056$

Hence the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3 rule.

Example 6: ***Calculate (up to 3 places of decimal)*** $\int_2^{10} \frac{dx}{1+x}$ ***by dividing the range into eight equal parts.*** **[Meerut 2001, 2007; B.C.A. (Kurukshetra) 2007, 2012]**

Solution: Dividing the whole range of integration *i.e.* (2, 10) into eight equal parts each of width $h = 1$, computed values of $y = \frac{1}{1+x}$ at each point of subdivision; the computed values are as follows:

x :	$x_0 = 2$	3	4	5	6	7	8	9	$10 = x_0 + 8h$
$y_x = \frac{1}{1+x}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$

By Simpson's '$\frac{1}{3}$' rule, we get

$$\int_{x_0}^{x_0+8h} y\,dx = \frac{h}{3}[y_{x_0} + y_{x_0+8h} + 4(y_{x_0+h} \dots + y_{x_0+7h})] + 2(y_{x_0+2h} + \dots + y_{x_0+6h})]$$

$$= \frac{h}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{1}{3}\left[\frac{1}{3} + \frac{1}{11} + 4\left\{\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right\} + 2\left\{\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right\}\right]$$

$$= 1.2996$$

Example 7: ***Show that*** $\int_0^1 \frac{dx}{1+x} = \log 2 = 0.69315.$

[B.C.A. (Agra) 2000]

Solution: Dividing the whole range (0, 1) into 10 equal parts making use of Simpson's '$\frac{1}{3}$' rule, we have the values of y at each point of sub-division are given below:

x	$y = \frac{1}{1+x}$
$x_0 = 0$	1
$x_0 + h = .1$	$\frac{1}{1.1} = .9090909$
$x_0 + 2h = .2$	$\frac{1}{1.2} = .8333333$
$x_0 + 3h = .3$	$\frac{1}{1.3} = .7692307$
$x_0 + 4h = .4$	$\frac{1}{1.4} = .7142857$
$x_0 + 5h = .5$	$\frac{1}{1.5} = .666666$
$x_0 + 6h = .6$	$\frac{1}{1.6} = .6250000$
$x_0 + 7h = .7$	$\frac{1}{1.7} = .5882352$
$x_0 + 8h = .8$	$\frac{1}{1.8} = .555555$
$x_0 + 9h = .9$	$\frac{1}{1.9} = .5263157$
$x_0 + 10h = 1$	$\frac{1}{2} = .500000$

$$\int_0^1 \frac{dx}{1+x} = \int_{x_0}^{x_0+10h} y\,dx = \frac{h}{3}[y_0 + y_{10} + 4(y_1 + \ldots + y_9) + 2(y_2 + \ldots + y_8)]$$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + 4\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) + 2\left(\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8}\right)\right]$$

$$= \frac{1}{30}[1.5 + 4 \times 3.45955 + 2 \times 2.72818]$$

$$= \frac{1}{30}[20.79456] = .693152 = .69315$$ upto five decimal places.

Again $\int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1 = \log 2.$

Hence, $\log 2 = \int_0^1 \frac{1}{1+x}\,dx = .69315.$

Example 8: ***Calculate*** $\int_0^{\pi/2} e^{\sin x}\,dx$, ***correct to four decimal places.***

[B.C.A. (I.P.U.) 2010]

Solution: Divide the range $(0, \pi/2)$ into three equal parts taking $h = \pi/6$ and compute the values of the function at each points of sub-division.

The values of the function are given below:

x :	0	$\pi/6$	$\pi/3$	$\pi/2$
$y_x = e^{\sin x}$:	1	1.64872	2.45960	2.71828

By Simpson's '$\frac{3}{8}$' rule, we have

$$\int_0^{\pi/2} e^{\sin x}\,dx = \frac{3h}{8}[y_0 + y_3 + 3(y_1 + y_2)]$$

$$= \frac{3}{8} \cdot \frac{\pi}{6}[1 + 2.71828 + 3(1.64872 + 2.45960)]$$

$$= \frac{\pi}{16}[3.71828 + 12.32496] = \frac{\pi}{16}[16.04324] = 3.150722930.$$

Example 9: ***Find the value of*** $\log 2$ ***from*** $\int_0^1 \frac{x^2}{1+x^3}\,dx$, ***using Simpson's '***$\frac{1}{3}$***' rule, by dividing the range into four equal parts. Also find the error.***

[Agra 2003; Avadh 2003]

Solution: Divide the whole range (0, 1) into four equal parts taking $h = .25$. The values of the function $y = \frac{x^2}{1+x^3}$ at the point of sub-division are given below:

x :	0	.25	.50	.75	1.000
y_x :	0	.06153	.22222	.39560	.5000

By Simpson's '$\frac{1}{3}$' rule, we have

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \int_{x_0}^{x_0+4h} y\,dx = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{.25}{3}[0 + .500 + 4(.06153 + .39560) + 2(.22222)]$$

$$= \frac{.25}{3}[.500 + 1.82852 + .4444] = .23108.$$

Again actual value of the integral is

$$\int_0^1 \frac{x^2}{1+x^2}\,dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1 = \frac{1}{3}\log 2 = \frac{1}{3}(.69315) = .23105$$

Therefore the error $= .23105 - .23108 = -.00003$.

Example 10: ***Evaluate*** $\int_0^4 e^x\,dx$, ***by Simpson's rule, using the data*** $e = 2.72, e^2 = 7.39$, $e^3 = 20.09, e^4 = 54.60$ ***and compare it with the actual value.***

[Lucknow 2009; Avadh 2005]

Solution: Divide the whole range $(0, 4)$ into four equal parts by taking $h = 1$. By Simpson's '$\frac{1}{3}$' rule, we have

$$\int_0^4 e^x\,dx = \int_{x_0}^{x_0+4h} y\,dx = \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{3}[1 + 54.60 + 4(2.72 + 20.09) + 2(7.39)]$$

$$= \frac{1}{3}[55.60 + 91.24 + 14.78] = \frac{1}{3}[161.62] = 53.873.$$

The actual value of the integral is

$$\int_0^4 e^x\,dx = e^4 - e^0 = 54.60 - 1.00 = 53.60.$$

Example 11: ***Evaluate*** $\int_4^{5.2} \log_e x\,dx$, ***by***

(i) Simpson's '$\frac{1}{3}$'

(ii) '$\frac{3}{8}$' rule. After finding the true value of the integral, compare the errors in three cases.

[Lucknow 2008; Bhopal 2002; B.C.A. (Meerut) 2002, 2006]

Solution: Divide the whole range of integration *i.e.* $(4, 5.2)$ into six equal parts taking $h = .2$ and then compute the values of the function $f(x) = \log_e x$ for each of sub-division. These computed values are as follows:

x	$y = \log_e x$
$x_0 = 4.0$	1.38629436
$x_0 + h = 4.2$	1.43508453
$x_0 + 2h = 4.4$	1.48160454
$x_0 + 3h = 4.6$	1.52605630
$x_0 + 4h = 4.8$	1.56861592
$x_0 + 5h = 5.0$	1.60943791
$x_0 + 6h = 5.2$	1.64865863

(i) By Simpson's '$\frac{1}{3}$' rule, we have

$$\int_4^{5.2} \log_e x\, dx = \frac{h}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{.2}{3}[3.03495299 + 4(4.57056874) + 2(3.05022046)]$$

$$= 1.82784726.$$

(ii) By Simpson's '$\frac{3}{8}$' rule, we have

$$\int_4^{5.2} \log_e x\, dx = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3(.2)}{8}[3.03495299 + 3(6.09424290) + 2(1.52605630)]$$

$$= \frac{.3}{4}[24.37129429 1] = 1.82784707.$$

Hence the errors are

(i) Due to Simpson's '$\frac{1}{3}$' rule $= .00000015$.

(ii) Due to Simpson's '$\frac{3}{8}$' rule $= .00000034$.

Example 12: ***Calculate an approximate value of*** $\int_0^{\pi/2} \sin x\, dx$***, by (i) the Trapezoidal rule, (ii) Simpson's rule, using 11 ordinates.***

[B.C.A. (Lucknow) 2008]

Solution: Divide the range of integration into ten equal parts taking $h = \pi/20$ and then compute the values of the function $f(x) = \sin x$ for each point of sub-division. These computed values are as follows:

x	$y = \sin x$
$x_0 = 0$	.00000
$x_0 + h = \pi/20$	.15663
$x_0 + 2h = 2\pi/20$	.30902
$x_0 + 3h = 3\pi/20$	.45399
$x_0 + 4h = 4\pi/20$	.58778
$x_0 + 5h = 5\pi/20$	.70711
$x_0 + 6h = 6\pi/20$	.80900
$x_0 + 7h = 7\pi/20$	.89101
$x_0 + 8h = 8\pi/20$	.95106
$x_0 + 9h = 9\pi/20$	.98769
$x_0 + 10h = 10\pi/20$	1.00000

(i) By the Trapezoidal rule, we have

$$\int_0^{\pi/2} \sin x\, dx = \frac{h}{2}[y_0 + y_{10} + 2(y_1 + y_2 + \ldots + y_9)]$$

$$= \frac{(\pi/20)}{2}[1.00000 + 2(5.85311)] = \frac{\pi}{40}[12.70622] = .9981.$$

(ii) By Simpson's '$\frac{1}{3}$' Rule, we have

$$\int_0^{\pi/2} \sin x\, dx = \frac{h}{3}[y_0 + y_{10} + 4(y_1 + y_3 + \ldots + y_9) + 2(y_2 + y_4 + \ldots + y_8)]$$

$$= \frac{\pi}{60}[1.00000 + 4(3.19623) + 2(2.65688)]$$

$$= \frac{\pi}{60}[1.00000 + 12.78492 + 5.31376] = \frac{\pi}{60}[19.09868] = 1.0006.$$

Again the exact value of the integral

$$\int_0^{\pi/2} \sin x\, dx = \left[-\cos x\right]_0^{\pi/2} = 1.0000$$

Therefore the error

(i) Due to Simpson's '$\frac{1}{3}$' rule $= -.0006$.

(ii) Due to Trapezoidal rule $= .0019$.

Example 13: ***Use Simpson's rule to prove that*** $\log_e 7$ ***is approximately*** 1.9587 ***using*** $\int_1^7 \frac{dx}{x}$. **[Lucknow 2005, 2009]**

Solution: Let us divide the range (1, 7) into six equal parts, each of width 1 unit. Hence $h = 1$. The values of y for each point of sub-division are as given below:

$x:$	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$	$x_4 = 5$	$x_5 = 6$	$x_6 = 7$
$y:$	$y_0 = 1$	$y_1 = 1/2$	$y_2 = 1/3$	$y_3 = 1/4$	$y_4 = 1/5$	$y_5 = 1/5$	$y_6 = 1/7$

Hence using Simpson's '1/3' rule, we obtain

$$\int_1^7 \frac{dx}{x} = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[1.1428571 + 3.6666665 + 1.0666667\right] = \frac{1}{3} \times 5.8761905$$

$$= 1.9587302 = 1.9587.$$

Also, the exact value of $\int_1^7 \frac{dx}{x} = \left[\log x\right]_0^7 = \log_e 7$.

$\therefore$ $\log_e 7 = 1.9587.$

Example 14: ***The velocity v (km / min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:***

t	0	2	4	6	8	10	12	14	16	18	20
v	0	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution: If s (km) be the distance covered in t (min), then we have

$$\frac{ds}{dt} = v$$

$\Rightarrow$ $$\left[s\right]_0^{20} = \int_0^{20} v\, dt = \frac{h}{3}[X + 4\,.\rho + 2\,.E],\ \text{by Simpson's rule}$$

Here $h = 2, v_0 = 0, v_1 = 10, v_2 = 18, v_3 = 25$ etc.

$\Rightarrow$ $$X = v_0 + v_{10} = 0 + 0 = 0$$

$$\rho = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$

$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance

$$= \left[s\right]_0^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72) = 309.33 \text{ km.}$$

Example 15: ***A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines x = 0 and a curve through the points with the following co-ordinates:***

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Solution: Here $h = 0.25, y_0 = 1, y_1 = 0.9896, y_2 = 0.9589$ etc.

Then the required of the solid generated

$$= \int_0^1 \pi\, y^2\, dx = \pi\,.\frac{h}{3}[(y_0^2 + y_4^2) + 4\,(y_1^2 + y_3^2) + 2\,y_2^2]$$

$$= \frac{0.25\,\pi}{3}[\{1 + (0.8415)^2\} + 4\,\{(0.9896)^2 + (0.9089)^2\} + 2\,(0.9589)^2]$$

$$= 0.2618\,(10.7687) = 2.8192.$$

Example 16: ***Evaluate*** $\int_{0.5}^{0.7} x^{1/2}\, e^{-x}\, dx$, ***approximately by using a suitable formula.***

[B.C.A. M.D.U. (Rohtak) 2003, 2005, 2010; B.C.A. (Rohilkhand) 2001, 2004, 2012]

Solution: Let us divide the range of integration (.5, .7) into 4 equal parts each of width $\frac{.7-.5}{4} = .05; h = .05$. The values of y for each point of sub-division are as given below:

x	$y = x^{1/2}\, e^{-x}$
$x_0 = .50$	.4288818
$x_0 + h = .55$	.4278774
$x_0 + 2h = .60$	.4251076
$x_0 + 3h = .65$	.4208867
$x_0 + 4h = .70$	.4154730

Whence using Simpson's '1/3' rule, we obtain

$$\int_{.5}^{.7} x^{1/2}\, e^{-x}\, dx = \frac{h}{3}\left[y_0 + y_4 + 4\,(y_1 + y_3) + 2\,y_2\right]$$

$$= \frac{.05}{3}\left[.8443548 + 3.3950564 + .8502152\right] = .0848271.$$

Example 17: ***If the third order differences are constant, prove that***

$$\int_0^2 \boldsymbol{U_x\, dx} = \frac{\mathbf{1}}{\mathbf{24}}\,[\boldsymbol{U_{-1/2} + 23U_{1/2} + 23U_{3/2} + U_{5/2}}].$$

Solution: The third order differences are given to be constant, so we may take U_x as a polynomial of degree 3. *i.e.*

$$U_x = a + bx + cx^2 + dx^3 \qquad ...(1)$$

Now, changing the scale to two times, we have to prove that

$$\int_0^4 U_x\, dx = \frac{1}{12}\,[U_{-1} + 23\,U_1 + 23\,U_3 + U_5].$$

Shifting the origin to -2, we have to prove that

$$\int_{-2}^{2} U_x\, dx = \frac{1}{12}\,[U_{-3} + 23\,U_{-1} + 23\,U_1 + U_3]. \qquad ...(2)$$

Now, L.H.S. of (2) $= \int_{-2}^{2} U_x\, dx = \int_{-2}^{2} (a + bx + cx^2 + dx^3)\, dx$

$$= \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4}\right]_{-2}^{2} = 2\left(2a + \frac{8c}{3}\right) = 4a + \frac{16c}{3}$$

and R.H.S. of (2) $= \frac{1}{12}\,[(U_{-3} + U_3) + 23\,(U_{-1} + U_1)]$

$$= \frac{1}{12}\,[2\,(a + 9c) + 23 \times 2\,(a + c)] = \frac{1}{6}\,(24a + 32c) = 4a + \frac{16c}{3} = \textbf{L.H.S.}$$

Example 18: ***If third differences are constant, prove that***

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{3}\,[f(0) + f(1/\sqrt{2}) + f(-1/\sqrt{2})].$$

[B.C.A. (Bhopal) 2010; B.C.A. (Avadh) 2004, 2010, 2012]

Solution: The third differences are given to be constant, so we consider $f(x)$ as a polynomial of degree 3

$$\Rightarrow \qquad f(x) = a + bx + cx^2 + dx^3 \qquad ...(1)$$

Now

$$\text{L.H.S.} = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (a + bx + cx^2 + dx^3)\,dx$$

$$= \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4}\right]_{-1}^{1} = 2\left(a + \frac{c}{3}\right)$$

and $\text{R.H.S.} = \dfrac{2}{3}\left[a + \left(a + \dfrac{b}{\sqrt{2}} + \dfrac{c}{2} + \dfrac{d}{2\sqrt{2}}\right) + \left(a - \dfrac{b}{\sqrt{2}} + \dfrac{c}{2} - \dfrac{d}{2\sqrt{2}}\right)\right]$, with the help of (1)

$$= \frac{2}{3}(3a + c) = 2\left(a + \frac{c}{3}\right) = \textbf{L.H.S.}$$

Example 19: ***Prove Simpson's formula***

$$\int_{a}^{b} f(x)\,dx = \frac{b-a}{6n}\,[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + f(x_{2n})],$$

where $x_0 = a,\ x_{2n} = b$ ***and use it to evaluate***

$\displaystyle\int_0^2 \frac{dx}{x}$ ***and give estimates of error for*** $n = 1$ ***and*** 2 ***given that*** $\log_e 2 = 0.69315$.

Solution: We know that Simpson's '$\frac{1}{3}$' rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[f(x_0) + f(x_0 + nh) + 4\{f(x_0 + h) + f(x_0 + 3h) + \ldots\}$$
$$+ 2\{f(x_0 + 2h) + f(x_0 + 4h) + \ldots\}].$$

Now putting $x_0 = a, n = 2n, x_0 + 2nh = x_{2n} = b$ and $h = \dfrac{b-a}{2n}$, we get the required form of Simpson's rule.

When $n = 1$, the whole range $(1, 2)$, is divided into two equal parts by three ordinates x_0, x_1, x_2. Thus the above formula reduces to

$$\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_2} f(x)\,dx, \qquad \text{where} \quad f(x) = \frac{1}{x}, x_2 = 2, x_0 = 1$$

$$= \frac{b-a}{6n}\,[f(x_0) + 4f(x_1) + f(x_2)]$$

$$= \frac{2-1}{6 \times 1}\left\{\frac{1}{1} + \frac{4}{(3/2)} + \frac{1}{2}\right\} = \frac{1}{6} \times \frac{25}{6} = \frac{25}{36} = .69444.$$

Again, when $n=2$, divide the whole range $(1,2)$ into four equal parts by the ordinates $x_0 = 1, x_1, x_2, x_3, x_4 = 2$. Thus we have

$$\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_4} f(x)\,dx = \frac{b-a}{6n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= \left[\frac{1}{12} + 4\,\frac{1}{(5/4)} + 2\,.\,\frac{1}{(3/2)} + 4\,.\,\frac{1}{(7/4)} + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left[1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left[\frac{210 + 672 + 280 + 480 + 105}{210}\right] = \frac{1}{12}\left(\frac{1347}{210}\right) = .69325\,.$$

The exact value is $\int_1^2 \frac{dx}{x} = \left[\log x\right]_1^2 = \log 2 = .69315.$

Therefore the error when $n=1$ is $.69315 - .69444 = -.00129$ and when $n=2$ is $.69315 - .69325 = -.0001$.

Example 20: ***If*** $f(x) = a + bx + cx^2$ ***prove that***

$$\int_1^2 f(x)\,dx = \frac{1}{12}[f(0) + 22f(2) + f(4)].$$

Solution: Since $f(x)$ is a polynomial of second degree in x, so that in this case third and higher differences are zero, also here we have $h=2$.

Again $$f(x) = f\left(\frac{x}{2}.2\right) = f\left(\frac{x}{2}.h\right) = E^{x/2} f(0)$$

$$= (1+\Delta)^{x/2} f(0) = f(0) + \frac{x}{2}\Delta f(0) + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)}{2\,!}\Delta^2 f(0) \qquad \text{(other terms vanish)}$$

$$\therefore \qquad \int_1^3 f(x)\,dx = \int_1^3 \left[f(0) + \frac{x}{2}\Delta f(0) + \frac{\frac{x^2}{4} - \frac{x}{2}}{2}\Delta^2 f(0)\right] dx$$

$$= \left[xf(0) + \frac{x^2}{4}\Delta f(0) + \frac{\frac{x^3}{12} - \frac{x^3}{4}}{2}\Delta^2 f(0)\right]_1^3$$

$$= 2f(0) + 2\Delta f(0) + \frac{1}{12}\Delta^2 f(0)$$

$$= 2f(0) + 2[f(2) - f(0)] + \frac{1}{12}[f(4) - 2f(2) + f(0)]$$

$$= \frac{1}{12}[f(0) + 22f(2) + f(4)].$$

Example 21: ***If*** $U_x = a + bx + cx^2$, ***prove that*** $\int_1^3 U_x\,dx = 2U_2 + \frac{1}{12}(U_0 - 2U_2 + U_4)$ ***and find approximation to*** $\int_{-1/2}^{1/2} e^{-(x^2/10)}\,dx$.

[B.C.A. (Bhopal) 2002, 2007, 2011, 2012]

Solution: Shifting the origin to -2, we have to prove that

$$\int_{-1}^{1} U_x\,dx = 2U_0 + \frac{1}{12}(U_{-2} - 2U_0 + U_2). \qquad \text{...(1)}$$

Now L.H.S. of (1) $= \int_{-1}^{1} U_x\,dx = \int_{-1}^{1} (a + bx + cx^2)\,dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_{-1}^{1} = 2\left(a + \frac{c}{3}\right)$.

Again $\qquad U_x = a + bx + cx^2$

$\therefore \qquad U_0 = a; U_{-2} = a - 2b + 4c; U_2 = a + 2b + 4c$

$\therefore \qquad$ R.H.S. of (1) $= 2U_0 + \frac{1}{12}(U_{-2} - 2U_0 + U_2) = 2a + \frac{1}{12}(2a + 8c - 2a)$

$$= 2\left(a + \frac{c}{3}\right) = \text{L.H.S.}$$

Now changing the scale to $\frac{1}{2}$ in (1), we get

$$\int_{-1/2}^{1/2} U_x\,dx = \frac{1}{2}\left[2U_0 + \frac{1}{12}(U_{-1} - 2U_0 + U_1)\right]$$

$$= U_0 + \frac{1}{24}(U_{-1} - 2U_0 + U_1).$$

Now putting $U_x = e^{-(x^2/10)}$, we get

$$\int_{-1/2}^{1/2} e^{-(x^2/10)}\,dx = 1 + \frac{1}{24}\left(e^{-(1/10)} + e^{-(1/10)} - 2\right)$$

$$= 1 + \frac{1}{12}\left(e^{-(1/10)} - 1\right)$$

Example 22: ***Obtain the approximate quadrature formula.***

(i) $\int_{-1/2}^{3/2} f(x)\,dx = \frac{1}{24}[27f(0) + 17f(1) + 5f(2) - f(3)]$

(ii) $\int_{-1}^{1} f(x)\,dx = \frac{1}{12}[13\{f(1) + f(-1)\} - \{f(3) + f(-3)\}]$

(iii) $\int_{0}^{2} f(x)\,dx = \frac{1}{24}[f(-\frac{1}{2}) + 23f(\frac{1}{2}) + 23f(\frac{3}{2}) + f(\frac{5}{2})]$, ***if third differences are constant.***

(iv) $\int_{-1/2}^{1/2} f(x)\,dx = \frac{1}{2}\{f(-\frac{1}{2}) + f(\frac{1}{2})\} - \frac{1}{24}\{\Delta f(-\frac{3}{2}) - \Delta f(\frac{1}{2})\}$.

Solution: (i) Here we are to obtain a formula in terms of $f(0), f(1), f(2), f(3)$ so the third differences are considered to be constant.

$$\therefore \quad \int_{-1/2}^{3/2} f(x)\,dx = \int_{-1/2}^{3/2} \left[f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) \right] dx$$

$$= \left[x f(0) + \frac{x^2}{2}\Delta f(0) + \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Delta^2 f(0) + \frac{1}{6}\left(\frac{x^4}{4} - x^3 + x^2\right) + \Delta^3 f(0) \right]_{-1/2}^{3/2}$$

$$= 2f(0) + \Delta f(0) + \frac{1}{12}\Delta^2 f(0) + \frac{1}{6}\left(-\tfrac{1}{4}\right)\Delta^3 f(0)$$

$$= 2f(0) + f(1) - f(0) + \frac{1}{12}[f(2) - 2f(1) + f(0)] - \frac{1}{24}[f(3) - 3f(2) + 3f(1) - f(0)]$$

$$= \frac{1}{24}[27f(0) + 17f(1) + 5f(2) - f(3)].$$

(ii) In this case we have to derive a formula in terms of $f(-3), f(-1), f(1), f(3)$; obviously $x_0 = -3$ and $h = 2$.

Now $$f(x) = f(-3+3+x) = f\left(-3 + \frac{3+x}{2}.2\right)$$

$$= f\left(-3 + \frac{3+x}{2}h\right) \qquad [\text{form } f(a+nh)]$$

$$\therefore \quad f(x) = E^{(3+x)/2} f(-3)$$

$$= \left[f(-3) + \frac{x+3}{2}\Delta f(-3) + \frac{\left(\frac{x+3}{2}\right).\left(\frac{x+3}{2}-1\right)}{2}\Delta^2 f(-3) + \frac{\left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}-1\right)\left(\frac{x+3}{2}-2\right)}{6}\Delta^3 f(-3) \right]$$

considering upto third differences.

$$\therefore \quad \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \left[f(-3) + \frac{x+3}{2}\Delta f(-3) + \frac{\left(\frac{x+3}{2}\right).\left(\frac{x+3}{2}-1\right)}{2}\Delta^2 f(-3) + \frac{\left(\frac{x+3}{2}\right)\left(\frac{x+3}{2}-1\right)\left(\frac{x+3}{2}-2\right)}{6}\Delta^3 f(-3) \right] dx$$

$$= \left[x\, f(-3) + \frac{1}{2}\left(\frac{x^3}{2} + 3x\right) \Delta f(-3) \right.$$

$$+ \frac{1}{2}\left\{ \frac{\frac{x^3}{3} + \frac{6x^2}{2} + 9x}{4} - \frac{\frac{x^2}{2} + 3x}{2} \right\} \Delta^2 f(-3)$$

$$\left. + \frac{1}{6}\left\{ \frac{\frac{x^4}{4} \cdot \frac{9x^3}{3} + 27\frac{x^2}{2} + 27x}{8} - \frac{3}{4}\left(\frac{x^3}{3} + 6x^2 + 9x\right) + \frac{3}{2}\left(\frac{x^3}{2} + 3x\right) \right\} \Delta^3 f(-3) \right]_{-1}^{1}$$

$$= 2 f(-3) + \frac{1}{2} 6 \Delta f(-3) + \frac{1}{2}\left(\frac{5}{3}\right) \Delta^2 f(-3) + \frac{1}{6}\left(-\frac{1}{2}\right) \Delta^3 f(-3)$$

$$= 2 f(-3) + 3[f(-1) - f(-3)] + \frac{5}{6}[f(1) - 2f(-1) + f(-3)]$$

$$- \frac{1}{12}[f(3) - 3f(1) + 3f(-1) - f(-3)]$$

$$= \frac{1}{12}[-f(-3) + 13f(-1) + 13f(1) - f(3)]$$

$$= \frac{1}{12}[13\{f(-1) + f(1)\} - \{f(3) + f(-3)\}].$$

(iii) In this case $x_0 = \frac{1}{2}$ and $h = 1$.

$$\therefore \quad f(x) = f\left(-\frac{1}{2} + x + \frac{1}{2}\right) \qquad \text{[from } f(x_0 + nh)\text{]}$$

$$= E^{x+(1/2)} f\left(-\frac{1}{2}\right) = (1 + \Delta)^{x+(1/2)} f\left(-\frac{1}{2}\right)$$

$$= f\left(-\frac{1}{2}\right) + \left(x + \frac{1}{2}\right) \Delta f\left(-\frac{1}{2}\right) + \frac{\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)}{2} \Delta^2 f\left(-\frac{1}{2}\right)$$

$$+ \frac{\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{2}\right)}{2} \Delta^3 f\left(-\frac{1}{2}\right) \text{ considering upto third differences.}$$

$$\therefore \int_0^2 f(x)\, dx = \int_0^2 \left[f\left(-\frac{1}{2}\right) + \left(x + \frac{1}{2}\right) \Delta f\left(-\frac{1}{4}\right) + \left(\frac{x^2 - \frac{1}{4}}{2}\right) \Delta^2 f\left(-\frac{1}{2}\right) \right.$$

$$\left. + \frac{1}{6}\left(x^3 - \frac{x}{4} - \frac{3x^2}{2} + \frac{3}{8}\right) \Delta^3 f\left(-\frac{1}{2}\right) \right] dx$$

$$= \left[xf\left(-\frac{1}{2}\right) + \left(x^2 + \frac{1}{2}x)\right) \Delta f\left(-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{4}x\right)\Delta^2 f\left(-\frac{1}{2}\right)\right.$$

$$\left. + \frac{1}{6}\left(\frac{x^4}{4} - \frac{x^2}{8} - \frac{3x^3}{6} + \frac{3x}{8}\right)\Delta^3 f\left(-\frac{1}{2}\right)\right]_0^2$$

$$= 2f\left(-\frac{1}{2}\right) + 3\Delta f\left(-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{13}{6}\right)\Delta^2 f\left(-\frac{1}{2}\right) + \frac{1}{8}\left(\frac{1}{4}\right)\Delta^3 f\left(-\frac{1}{2}\right)$$

$$= 2f\left(-\frac{1}{2}\right) + 3\left[f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right)\right] + \frac{13}{12}\left[f\left(\frac{1}{2}\right) - 2f\left(\frac{1}{2}\right) + f\left(-\frac{1}{2}\right)\right]$$

$$+ \frac{1}{24}\left[f\left(\frac{5}{2}\right) - 3f\left(\frac{3}{2}\right) - 3f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{24}\, f\left(-\frac{1}{2}\right) + 23f\left(\frac{1}{2}\right) + 23f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right)\Bigg]$$

$$\int_{-1/2}^{1/2} f(x)\,dx = \frac{1}{2}\left\{f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right\} + \frac{1}{24}\left\{f\left(-\frac{1}{2}\right) - f\left(-\frac{3}{2}\right) - f\left(\frac{3}{2}\right) + f\left(\frac{1}{2}\right)\right\}$$

$$= \frac{1}{24}\left[13f\left(-\frac{1}{2}\right) + 13f\left(\frac{1}{2}\right) - f\left(-\frac{3}{2}\right) - f\left(\frac{3}{2}\right)\right].$$

(iv) In this case $h = 1, x_0 = -\frac{3}{2}$.

Now $f(x) = f\left(-\frac{3}{2} + x + \frac{3}{2}\right)\Big]$ [From $f(x_0 + nh)$].

$$\therefore \quad \int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{1/2} \left[f\left(-\frac{3}{2}\right) + \left(x + \frac{3}{2}\right)\Delta f\left(-\frac{3}{2}\right)\right.$$

$$\left. + \frac{\left(x + \frac{3}{2}\right)\left(x + \frac{1}{2}\right)}{2\,!}\Delta^2 f\left(-\frac{3}{2}\right) + \frac{\left(x + \frac{3}{2}\right)\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)}{3\,!}\Delta^3 f\left(-\frac{3}{2}\right)\right] dx$$

$$= \left[xf\left(-\frac{3}{2}\right) + \left(\frac{x^2}{2} + \frac{3x}{2}\right)\Delta f\left(-\frac{3}{2}\right) + \frac{1}{2}\left(\frac{x^3}{3} + \frac{2x^2}{2} + \frac{3x}{4}\right)\Delta^2 f\left(-\frac{3}{2}\right)\right.$$

$$\left. + \frac{1}{6}\left(\frac{x^4}{4} - \frac{x^2}{8} + \frac{3}{2} \cdot \frac{x^3}{3} - \frac{3x}{8}\right)\Delta^3 f\left(-\frac{3}{2}\right)\right]_{-1/2}^{1/2}$$

$$= f\left(\frac{3}{2}\right) + \frac{3}{2}\Delta f\left(-\frac{3}{2}\right) + \frac{1}{2}\left[\frac{10}{12}\right]\Delta^2 f\left(-\frac{3}{2}\right) + \frac{1}{6}\left(-\frac{1}{4}\right)\Delta^3 f\left(-\frac{3}{2}\right)$$

$$= f\left(-\frac{3}{2}\right) + \frac{3}{2}\left[f\left(-\frac{1}{2}\right) - f\left(-\frac{3}{2}\right)\right] + \frac{5}{12}\left[f\left(\frac{1}{2}\right) - 2f\left(-\frac{1}{2}\right) + f\left(-\frac{3}{2}\right)\right]$$

$$- \frac{1}{24}\left[f\left(\frac{3}{2}\right) - 3f\left(\frac{1}{2}\right) + 3f\left(-\frac{1}{2}\right) - f\left(-\frac{3}{2}\right)\right]$$

$$= \frac{1}{24}\left[-f\left(-\frac{3}{2}\right) + 13f\left(-\frac{1}{2}\right) + 13f\left(\frac{1}{2}\right) - f\left(\frac{3}{2}\right)\right].$$

Problem Set

1. Calculate the approximate value of $\int_{-3}^{3} x^4 \, dx$ by using (i) Trapezoidal rule, (ii) Simpson's rule, by dividing the range in six equal parts.
[B.C.A. (Kanpur) 2006]

2. Use Simpson's rule dividing the range into ten equal parts, to show that $\int_0^1 \frac{\log(1+x^3)}{(1+x^2)} = 0.1730$. **[B.C.A. (Lucknow) 2002, 2008]**

3. Use Simpson's '$\frac{1}{3}$' rule to find $\int_0^6 \frac{dx}{(1+x)^2}$.

4. Evaluate $\int_3^5 \frac{4}{(2+x^2)} dx$ by dividing the range into eight equal parts.

5. Evaluate $\int_5^{12} \frac{dx}{x}$ by numerical methods.

6. Use Simpson's rule to prove that $\log_e 7$ is approximately 1.958 using $\int_0^7 \frac{dx}{x}$.

7. Evaluate $\int_0^1 x^2 \, dx$ by Simpson's '$\frac{1}{3}$' rule. **[B.C.A. (Agra) 2010]**

8. Find approximate value of $\int_0^2 \frac{dx}{x}$ by using Simpson's Rule.

9. Use Simpson's '$\frac{1}{3}$' rule to find value of $\int_1^5 f(x) \, dx$ given

x :	1	2	3	4	5
***f*(*x*) :**	10	50	70	80	100

10. Use Simpson's '$\frac{1}{3}$' rule to find the approximate area of the cross-section of a river 80 metres wide, the depth y (in metres) at a distance x from one bank being given by the following table:

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

[**Hint:** Area $= \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$

$= \frac{10}{3}[3 + 4(36) + 2(33)] = 710$ *sq.* metre]

11. Prove that if $f(x)$ is continuous function whose fifth differences are constant, then

$$\int_{-1}^{+1} f(x)\,dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\sqrt{(0.6)} + f\{-\sqrt{(0.6)}\}\right].$$

12. Obtain the approximate quadrature formula

$$\int_{0}^{n} f(x)\,dx = n\left[\frac{3}{8} f(0) + \frac{1}{24}\{19f(n) - 5f(2n) + f(3n)\}\right].$$

13. Evaluate $\int_{100}^{200} \frac{dx}{\log x}$ and compare with the exact value.

14. (i) If $f(x)$ is a polynomial of degree 2, prove that

$$\int_{0}^{1} f(x)\,dx = \frac{1}{12}[5f(0) + 8f(1) - f(2)].$$

(ii) Prove that $\int_{0}^{1} f(x)\,dx = \frac{1}{12}[5f(1) + 8f(0) - f(-1)]$.

15. Use Simpson's rule with two, four and ten intervals to evaluate $\int_{1}^{2} \frac{1}{x}\,dx = \log 2$ and compare with $\log 2 = 0.693147$.

Answers

1.	(i) 115 (ii) 98
3.	.8946
4.	26.716
5.	0.87546854
7.	34598
8.	.6932
9.	256.66
10.	710 sq. metre
12.	0.00457

❍❍❍

Unit

4

Chapter-8: Solution of Linear Equations

Unit-4

Chapter 8

Solution of Linear Equations

8.1 The Elimination Method by Gauss

We have seen that any system of linear algebraic equations can be solved by the use of determinants. If the order of the determinant is large, then the evaluations become tedious. To avoid these unnecessary computations, from the practical computation point of view, various mathematicians have tried to develop simpler and less time consuming procedures, and various methods for solving system of linear equations have been suggested.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \quad\ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ \quad\ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right] \qquad \ldots(1)$$

One of the most important methods is the so called **Gauss-Algorithm**, which is a systematic elimination method. In this method, for solving (1) we proceed step wise as follows:

First Step: *Elimination of* x_1 *form the second, third ,..., nth equation.* We assume here that the order of the equation and the order of the unknowns in each equations are such that $a_{11} \neq 0$. The variable x_1 can then be eliminated from the second equation by subtracting (a_{21}/a_{11}) times the first equation from the second equation, (a_{31}/a_{11}) times the first equation from the third equation etc. This gives new system as follows:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ \ldots \qquad a'_{22}x_2 + \ldots + a'_{2n}x_n = b'_2 \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ \qquad a'_{n2}x_2 + \ldots + a'_{nn}x_n = b'_n \end{array}\right] \qquad \ldots(2)$$

***Second Step**: Elimination of x_2 form the third ,...., nth equation (2).*

If the coefficient $a'_{22}, \ldots, a'_{nn}$ in (2) are not all zero, we may assume that the order of equation and the unknowns is such that $a'_{22} \neq 0$. Then we may eliminate x_2 from the third, fourth,..., nth equations of (2) by subtracting.

(a'_{33}/a'_{22}) times the second equation from the third equation, (a'_{42}/a'_{22}) times the second equation from the fourth equation etc. The fourth steps are now obvious. In the third step, we eliminate x_3 and in the fourth step, we eliminate x_4 etc.

By successive elimination, we arrive at a single equation in the unknown x_n which can be solved and substituting this in the preceding equation we obtain the value of $x_{(n-1)}$. In this manner, we find x_1 when the elimination is completed. Also when the elimination is complete, the system takes the form

$$\left.\begin{array}{llll} c_{11}x_1 + c_{12}\,x_2 + \ldots + c_{1n}x_n = d_1 \\ \quad\quad c_{22}\,x_2 \;+\ldots+\; c_{2n}\,x_n = d_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad c_{nn}\,x_n = d_n \end{array}\right] \textit{Triangular form.}$$

In this case, there exists a unique solution. The new coefficient matrix is an upper triangular matrix, the diagonal element c_{ii} are usually equal to 1.

8.2 Gauss-Seidel Iteration Method

[Lucknow 2008]

This is the modification of Jacobi's method. As before the system of equations:

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \quad \ldots(1)$$

can be written as

$$\left.\begin{array}{l} x = \dfrac{1}{a_1}(d_1 - b_1\,y - c_1\,z), \\ y = \dfrac{1}{b_2}(d_2 - a_2x - c_2z), \\ z = \dfrac{1}{c_3}(d_3 - a_3x - b_3\,y) \end{array}\right\} \quad \ldots(2)$$

Let us start with initial approximations x_0, y_0, z_0 for x, y, z respectively. Then substituting $y = y_0, z = z_0$ in the first of the equations (2), we get

$$x_1^{(1)} = \frac{1}{a_1}(d_1 - b_1\,y_0 - c_1\,z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2), we have

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1^{(0)} - c_2 z_0)$$

Next substituting $x = x_1^{(1)},\ y = y_1^{(1)}$ in the third of the equations (2), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1^{(1)} - b_3 y_1^{(1)}) \text{ etc.}$$

> **Note:** Jacobi and Gauss-Seidel methods converge for any choice of the initial approximations if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

SOLVED EXAMPLES

Example 1: ***Apply Gauss-Seidel iteration method to solve the equations***

$$\mathbf{20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25.}$$

[B.C.A. (Meerut) 2002, 2005, 2010; B.C.A.(Kanpur) 2006, 2012; B.C.A. (Avadh) 2008]

Solution: Select x, y, z from above equations respectively, we have

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{...(1)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{...(2)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{...(3)}$$

First Iteration: Putting $y = y_0, z = z_0$ is (1), we get

$$x_1^{(1)} = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1^{(1)}, z = z_0$ in (2), we have $y_1^{(1)} = \frac{1}{20}(-18 - 3x_1^{(1)} + z_0) = -1.0275$

Putting $x = x_1^{(1)},\ y = y_1^{(1)}$ in (3), we obtain $z_1^{(1)} = \frac{1}{20}(25 - 2x_1^{(1)} + 3y_1^{(1)}) = 1.0109$

Second Iteration: Putting $y = y_1^{(1)}, z = z_1^{(1)}$ in (1), we get

$$x_2^{(2)} = \frac{1}{20}(17 - y_1^{(1)} + 2z_1^{(1)}) = 1.0025$$

Putting $x = x_2^{(2)}, z = z_1^{(1)}$ in (2), we get $y_2^{(2)} = \frac{1}{20}(-18 - 3x_2^{(2)} + z_1^{(1)}) = -0.9998$

Again putting $x = x_2^{(2)}, y = y_2^{(2)}$ in (3), we have

$$z_2^{(2)} = \frac{1}{20}(25 - 2x_2^{(2)} + 3y_2^{(2)}) = 0.9998$$

***Third Iteration*:**

$$x_3^{(3)} = \frac{1}{20}(17 - y_2^{(2)} + 2z_2^{(2)}) = 1.0000$$

$$y_3^{(3)} = \frac{1}{20}(-18 - 3x_3^{(3)} + z_2^{(2)}) = -1.0000$$

$$z_3^{(3)} = \frac{1}{20}(25 - 2x_3^{(3)} + 3y_3^{(3)}) = 1.0000$$

Now the values in the 2nd and 3rd iterations are nearly the same,so we stop here. Hence the solution is $x = 1, y = -1, z = 1$.

***Example 2:* *Solve the equations*:**

$$\mathbf{10x_1 - 2x_2 - x_3 - x_4 = 3}$$

$$\mathbf{-2x_1 + 10x_2 - x_3 - x_4 = 15}$$

$$\mathbf{-x_1 - x_2 + 10x_3 - 2x_4 = 27}$$

$$\mathbf{-x_1 - x_2 - 2x_3 + 10x_4 = -9.}$$

By Gauss-Seidel Iteration Method. **[Rohilkhand 2009, 2012]**

Solution: Rewriting the given equations as given below:

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{...(1)}$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{...(2)}$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{...(3)}$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad \text{...(4)}$$

First Iteration:

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (1), we get $x_1^{(1)} = 0.3$

Putting $x_1^{(1)} = 0.3, x_3 = 0, x_4 = 0$ in (2), we obtain $x_2^{(1)} = 1.56$

Putting $x_1^{(1)} = 0.3, x_2^{(1)} = 1.56, x_4 = 0$ in (3), we obtain $x_3^{(1)} = 2.886$

Putting $x_1^{(1)} = 0.3, x_2^{(1)} = 1.56, x_3^{(1)} = 2.886$ in (4), we get $x_4^{(1)} = -0.1366$

Second Iteration:

Putting $x_2^{(1)} = 1.56, x_3^{(1)} = 2.886, x_4^{(1)} = -0.1368$ in (1), we obtain $x_1^{(2)} = 0.8869$

Putting $x_1^{(2)} = 0.8869, x_3^{(1)} = 2.886, x_4^{(1)} = -0.1368$ in (2), we obtain $x_2^{(2)} = 1.9523$

Putting $x_1^{(2)} = 0.8869, x_2^{(2)} = 1.9523, x_4^{(1)} = -0.1368$ in(3), we have $x_3^{(2)} = 2.9566$

Putting $x_1^{(2)} = 0.8896, x_2^{(2)} = 1.9523, x_3^{(2)} = 2.9566$ in (4), we get $x_4^{(2)} = -0.0248$.

Third Iteration:

Putting $x_2^{(2)} = 1.9523, x_3^{(2)} = 2.9566, x_4^{(2)} = -0.0248$ in (1), we have $x_1^{(3)} = 0.9836$

Putting $x_1^{(3)} = 0.9836, x_3^{(2)} = 2.9566, x_4^{(2)} = -0.0248$ in (2), we get $x_2^{(3)} = 1.9899$

Putting $x_1^{(3)} = 0.9836, x_2^{(3)} = 1.9899, x_4^{(2)} = -0.0248$ in (3), we get $x_3^{(3)} = 2.9924$

Then putting $x_1^{(3)} = 0.9836, x_2^{(3)} = 1.9800, x_3^{(3)} = 2.9924$ in (4), we get

$$x_4^{(3)} = -0.0042.$$

Fourth Iteration: Proceeding as above, we readily obtain

$$x_1^{(4)} = 0.9968, x_2^{(4)} = 1.9928, x_3^{(4)} = 2.9987, x_4^{(4)} = -0.0008.$$

Fifth Iteration:

$$x_1^{(5)} = 0.9994, x_2^{(5)} = 1.9997, x_3^{(5)} = 2.9997, x_4^{(5)} = -0.0001.$$

Sixth Iteration:

$$x_1^{(6)} = 0.9999, x_2^{(6)} = 1.9999, x_3^{(6)} = 2.9999, x_4^{(6)} = -0.0001$$

Hence, the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0.$

Example 3: Solve

$$\begin{aligned} 10x - 7y + 3z + 5u &= 6, \\ -6x + 8y - z - 4u &= 5, \\ 3x + y + 4z + 11u &= 2, \\ 5x - 9y - 2z + 4u &= 7. \end{aligned}$$

By Gauss's Elimination method.

Solution: The Given system can be written as

$$\left.\begin{aligned} x-0.7y+0.3z+0.5u&=0.6\\ -6x+8y-z-4u&=5\\ 3x+y+4z+11u&=2\\ 5x-9y-2z+4u&=7 \end{aligned}\right] \qquad \ldots(1)$$

The variable x can be eliminated from the second, third and fourth equations by subtracting (– 9) times the first equation from the second equation, 3 times the first equation from the third equation and 5 times the first equation from the fourth equation and thus new system becomes

$$\left.\begin{aligned} x-0.7y+0.3z+0.5u&=0.6\\ 3.8y+0.8z-u&=8.6\\ 3.1y+3.1z+9.5u&=0.2\\ -5.5y-3.5z+1.5u&=4 \end{aligned}\right\} \qquad \ldots(2)$$

In this system, coeff. of y is maximum (numerically) in fourth equation, so now, we shall permutate the second and fourth equation.

After that y is eliminated:

$$\left.\begin{aligned} x-0.7y+0.3z+0.5u&=0.6\\ y+0.6363z-0.27275u&=-0.72727\\ -1.61818z+0.03636u&=11.36364\\ 1.12727z+10.34545u&=2.45455 \end{aligned}\right\} \qquad \ldots(3)$$

Further elimination of z gives

$$\left.\begin{aligned} x-0.7y+0.3z+0.5u&=0.6\\ y+0.63636z-0.27273u&=-0.72727\\ z-0.02247u&=-7.02247\\ 10.3607947u&=10.37079 \end{aligned}\right\} \qquad \ldots(4)$$

These give $u=1,\ z=-7,\ y=4$ and $x=5$.

Example 4: *Solve*

$$x+\frac{1}{2}y+\frac{1}{3}z=1,$$
$$\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=0,$$
$$\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=0,$$

by Gauss's Elimination method.

Solution: Eliminate x from II$^{\text{nd}}$ and III$^{\text{rd}}$ with the help of I$^{\text{st}}$, we get

$$\left.\begin{aligned} x+\frac{1}{2}y+\frac{1}{3}z&=1\\ \frac{1}{12}y+\frac{1}{12}z&=-\frac{1}{2}\\ \frac{1}{12}y+\frac{4}{45}z&=-\frac{1}{3} \end{aligned}\right\} \qquad \ldots(1)$$

Now, we shall eliminate y from the third and fourth equations.

Subtracting second equation from third equation, we immediately get

$$\left.\begin{array}{r} x+\frac{1}{2}y+\frac{1}{3}z=1 \\ \frac{1}{12}y+\frac{1}{12}z=-\frac{1}{2} \\ -\frac{1}{180}z=-\frac{1}{6} \end{array}\right\} \quad \text{...(2)}$$

This gives $z=30,\ y=-36,\ x=9.$

Example 5: ***By Gauss's Elimination method, solve***

(*i*)

$$\mathbf{5x-y-2z=142,}$$

$$\mathbf{x-3y-z=-30}$$

$$\mathbf{2x-y-3z=-5.}$$

[B.C.A. (Bhopal) 2002, 2008, 2013]

(*ii*) $\mathbf{x+4y-z=-5,\ x+y-6z=-12;\ 3x-y-z=4.}$

Solution: (i) Elimination of x from second and third equations, we get

$$\left.\begin{array}{r} 5x-y-2z=\quad 142 \\ -\frac{14}{5}y-\frac{3}{5}z=-\frac{292}{5} \\ -\frac{5}{6}y-\frac{11}{5}z=-\frac{250}{5} \end{array}\right\} \quad \text{...(1)}$$

Further eliminating y from the third and fourth equations, we get

$$\left.\begin{array}{r} 5x-y-2z=142 \\ -\frac{14}{5}y-\frac{3}{5}z=-\frac{292}{5} \\ -\frac{29}{14}z=-\frac{275}{7} \end{array}\right\} \quad \text{...(2)}$$

This gives $x=39.2$ approx and $y=16.7$ approx.

(ii) We have

$$x+4y-z=-5 \quad \text{...(1)}$$

$$x+y-6z=-12 \quad \text{...(2)}$$

$$3x-y-z=4 \quad \text{...(3)}$$

Eliminating x from (2) and (3) with the help of (1)

[(2) – (1) and (3) – 3 (1)]; we get

$$-3y - 5z = -17 \qquad ...(4)$$

$$-13y + 2z = 19 \qquad ...(5)$$

To eliminate y let us operate $(5) - \frac{13}{4}[4]$

$$\Rightarrow \qquad \frac{71z}{3} = \frac{148}{3} \qquad ...(6)$$

Then substituting backwards, we get $z = (148/71)$ (from 6),

$$y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} \qquad \text{(from 4)},$$

and

$$x = -5 - 4\left(\frac{-81}{71}\right) + \frac{148}{71} = \frac{117}{71}.$$

Example 6:* *Solve the following system of equations by Gauss Seidel Iteration Method.

[B.C.A. (Agra) 2006, 2008, 2009, 2012]

$$\mathbf{27x + 6y - z = 85,}$$

$$\mathbf{6x + 15y + 2z = 72,}$$

$$\mathbf{x + y + 54z = 110.}$$

Solution: We solve each equation of the system of the unknown with the largest coefficient in terms of the remaining unknowns *viz.*,

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z).$$

and

$$z = \frac{1}{54}(110 - x - y).$$

Starting with $y = 0, z = 0$, we obtain

$$x = x^{(1)} = \frac{85}{27} = 3.15 = \text{first approximation.}$$

Now putting $x = 3.15, z = 0$ in the second equation, we get

$$y = (1/15)(72 - 18.90) = 3.54 = y^{(1)} = \text{first approx.}$$

Again putting $x = 3.15$, $y = 3.54$ in third equation, we get

$$z = z^{(1)} = (1/54)\,(110 - 3.15 - 3.54) = 1.91 = \text{first approx.}$$

Now, we proceed to obtain the second approximations.

$$x^{(2)} = (1/27)\,[85 - 6\,y^{(1)} + z^{(1)}] = (1/27)\,[85 - 21.24 + 1.91] = 2.43\,,$$

$$y^{(2)} = (1/15)\,[72 - 6x^{(2)} - 2z^{(1)}] = (1/15)\,[72 - 14.58 - 3.82] = 3.57,$$

$$z^{(2)} = (1/54)[110 - x^{(2)} - y^{(2)}] = (1/54)[110 - 2.43 - 3.57] = 1.926\,,$$

Similarly,

$$x^{(3)} = (1/27)\,[85 - 6\,y^{(2)} + z^{(2)}] = 2.426\,,$$

$$y^{(3)} = (1/15)[72 - 6x^{(3)} - 2z^{(2)}] = 3.572$$

$$z^{(3)} = (1/54)[110 - x^{(3)} - y^{(3)}] = 1.926$$

These values are sufficiently close to $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ receptively.

Hence the values 2.426, 3.572, 1.926 can be considered as the solution of the Given System.

Problem Set

1. Apply Gauss's Elimination Method to solve the following system:

(i)
$$16x_1 - 19x_2 + 40x_3 - 17x_4 = 89,$$
$$82x_1 + 42x_2 + 32x_3 - 8x_4 = 73,$$
$$55x_1 - 35x_2 + 45x_3 - 25x_4 = 67,$$
$$7x_1 + 19x_2 + 33x_3 + 85x_4 = 111.$$

(ii)
$$5x_1 - x_2 - 2x_3 = 142,$$
$$x_1 - 3x_2 - x_3 = -31,$$
$$2x_1 - x_2 - 3x_3 = 5.$$

2. Apply Gauss-Seidel Iteration Method to solve
$$10x + y + z = 12,$$
$$2x + 10y + z = 13,$$
$$2x + 2y + 10z = 14.$$

Unknown	**1**	**2**	**3**	**4**
x	1.20	90	1.00	1.00
y	1.06	1.00	1.00	
z	0.95	1.00	1.00	

3. Solve by Gauss-Elimination Method
$$2x + y + 4z = 12,$$
$$8x - 3y + 2z = 23,$$
$$4x + 11y - z = 33.$$
[B.C.A. (Lucknow) 2007, 2011]

4. Solve the system of equations
$$2x - 3y + 10z = 3, \; -x + 4y + 2z = 20, \; 5x + 2y + z = -12$$
By Gauss-Elimination Method.

5. Solve the following system of Linear equations, using Gauss-Elimination Method:
$$x_1 + 6x_2 + 3x_3 = 6$$
$$2x_1 + 3x_2 + 3x_3 = 117$$
$$4x_1 + x_2 + 2x_3 = 283.$$
[B.C.A. (Kanpur) 2010]

6. Using Gauss-Seidel Iterative Method and the starting solution $x_1 = x_2 = x_3 = 0$, determine the solution of the following system of equations in two iterations.

$$10x_1 - x_2 - x_3 = 8$$

$$x_1 + 10\,x_2 + x_3 = 12$$

$$x_1 - x_2 + 10x_3 = 10$$

Compare the approximate solution with the exact solution. **[B.C.A. (Agra) 2003]**

7. Using Gauss-Seidel Iterative Method, find the solution of the following system:

$$4x_1 - x_2 + 8x_3 = 26$$

$$5x_1 + 2x_2 - x_3 = 6$$

$$x_1 - 10x_2 + 2x_3 = -13.$$

upto three iterations.

with initial values (4.67, 7.62, 9.05) upto two iterations.

8. Solve the following system of equations by Gauss's Elimination Method:

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$

$$-6x_1 + 8x_2 - x_3 - 4x_4 = 5$$

$$3x_1 + x_2 + 4x_3 + 11x_4 = 2$$

$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7$$

[B.C.A. (I.P.U.) Delhi 2002]

9. Using Gauss-Seidel Iteration Method find the solution, correct to three decimal places of the Linear system

$$7x_1 + 52x_2 + 13x_3 = 104$$

$$3x_1 + 8x_2 + 29x_3 = 71$$

$$83x_1 + 11x_2 - 4x_3 = 95$$

with $(x_1^0, x_2^0, x_3^0) = (1.145, 1.846, 1.821)$ upto two iterations.

10. Solve the Given System of equations using Gauss Elimination Method with partial pivoting:

$$x - y + 3z = 13; \quad 4x - 2y + z = 15; \quad -3x - y + 4z = 8.$$

[B.C.A. (Meerut) 2004]

Answers

1.	(i) $x_1 = -1.339, x_2 = 2.5, x_3 = 0.288, x_4 = -8.61$ (ii) $x_1 = 39.378, x_2 = 17.17, x_3 = 18.86$ approx. (iii) $x_1 = 2, x_2 = -1, x_3 = 3$
2.	$x = 1.163,\ y = 1.14, z = -0.77$
3.	3.36, 1.8, 0.8
4.	$x = -4,\ y = 3, z = 2$
5.	$x_1 = 72, x_2 = -13, x_3 = 4$
6.	$x_1 = 1, x_2 = 1, x_3 = 1$
7.	$x_1 = 1, x_2 = 2, x_3 = 3$
8.	$x_1 = 5, x_2 = 4, x_3 = -7, x_4 = 1$
9.	$x = 1.06,\ y = 1.37, z = 1.96$
10.	$x = 2,\ y = -2, z = 3$

❍❍❍

Unit

5

Chapter-9: Solution of Differential Equations

Unit-5

Chapter 9

Solution of Differential Equations

9.1 *Picard's Method of Successive Approximations

[B.C.A. (Avadh) 2004; B.C.A. (Agra) 2003, 2005; B.C.A. (Bhopal) 2000, 2001, 2008, 2013; B.C.A. (Rohilkhand) 2008, 2012]

Consider the following differential equation:

$$\frac{dy}{dx} = f(x, y); \text{initial condition is that } y = y_0 \text{ at } x = x_0.$$

Integrating the above equation between the limits x_0 and x, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y - y_0 = \int_{x_0}^{x} f(x, y)\, dx,$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx. \qquad \ldots(1)$$

For a first approximation, we replace y and y_0 in $f(x, y)$; for a second, we replace it by the first approximation, for a third by the second and so fourth.

Now first approximation, $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx,$ $\qquad \ldots(2)$

Second approximation, $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx;$ $\qquad \ldots(3)$

… … … … …

nth approximation $\quad y^{(n)} = y_0 + \int_{x_0}^{x} f\,(x, y^{(n-1)})\, dx,$ with $y^{(0)} = y_0$

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*Picard was a distinguished professor of Mathematics at the University of Paris famous for his researches on the theory of functions.

As a matter of fact, the process is stopped when the two values of y *viz.* $y^{(n-1)}$ and $y^{(n)}$ are sensible the same *i.e.* they are same to the desired degree of accuracy.

Picard's method is of considerable theoretical value. In general, it is unsatisfactory as a practical means of approximation because of difficulties which arise in performing the necessary integrations.

Note: Each approximation gives a better approximation of the required solution than the preceding one.

SOLVED EXAMPLES

Example 1: ***Find the solution of*** $\frac{dy}{dx} = 1 + xy$ ***which passes through*** $(0, 1)$ ***in the interval*** $(0, 0.5)$ ***such that the value of y is correct to three decimal places (use the whole interval as one interval only). Take*** $h = 0.1$.

[B.C.A. (Meerut) 2010; B.C.A. (Kashi Vidhyapeeth) 2007, 2010, 2012; B.C.A. (Agra) 2002, 2010]

Solution: We have $\frac{dy}{dx} = 1 + xy$; giving $f(x, y) = 1 + xy$.

and $y = 1$ at $x_0 = 0$; giving $y_0 = 1$ and $x_0 = 0$.

Now first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 1 + \int_0^x \{1 + (xy_0)\}\, dx$$

$$= 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2} \qquad \ldots(1)$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = y_0 + \int_0^x \{1 + xy^{(1)}\}\, dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} \qquad \ldots(2)$$

Third approximation,

$$y^{(3)} = 1 + \int_0^x \{1 + (xy^{(2)})\}\, dx$$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right\} dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \qquad \ldots(3)$$

When $x = 0$, $y = 1$ *i.e.*, $y^{(0)} = 1$.

When $x = 0.1$, $y^{(1)} = 1.105$, $y^{(2)} = 1.10533$.

But we want accuracy up to 3 decimal places.

Therefore when $x = 0.1,\ y = 1.105$.

When $x = 0.2,\ y^{(1)} = 1.202,\ y^{(2)} = 1.2027$.

Therefore when $x = 0.2,\ y = 1.203$

When $x = 0.3;\ y^{(1)} = 1.345,\ y^{(2)} = 1.35501,\ y^{(3)} = 1.3551$.

Therefore when $x = 0.3,\ y = 1.355$.

When $x = 0.4;\ y^{(1)} = 1.48,\ y^{(2)} = 1.50453,\ y^{(3)} = 1.5052$.

Therefore when $x = 0.4,\ y = 1.505$.

When $x = 0.5;\ y^{(1)} = 1.625,\ y^{(2)} = 1.6744725$ and $y^{(2)} = 1.6765$

Therefore, we have $y = 1.677$ when $x = 0.5$

Now we tabulate the results as obtained above.

x	0	0.1	0.2	0.3	0.4	0.5
y	1.000	1.105	1.203	1.355	1.505	1.677

Thus we have solved numerically the given differential equation for

$$x = 0, 0.1, 0.2, 0.3, 0.4, 0.5.$$

Example 2: ***Use Picard's method to approximate*** y ***when*** $x = 0.2$, ***given that*** $y = 1$ ***when*** $x = 0$ ***and*** $\frac{dy}{dx} = x - y$.
[B.C.A. (Meerut) 2012; B.C.A. (Agra) 2003; B.C.A. (Avadh) 2003]

Solution: We have $f(x, y) = x - y,\ x_0 = 0,\ y_0 = 1$

Now first approximation, $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 1 + \int_0^x (x-1)\,dx = \frac{x^2}{2} - x + 1$.

Second approximation, $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx = 1 + \int_0^x \{x - (y)^{(1)}\}\,dx$

$$= 1 + \int_0^x \left(x - \frac{x^2}{2} + x - 1\right) dx = -\frac{x^3}{6} + x^2 - x + 1.$$

Third approximation, $y^{(3)} = 1 + \int_0^x f(x, y^{(2)})\,dx = 1 + \int_0^x (x - y^{(2)})\,dx$

$$= 1 + \int_0^x \left(x + \frac{x^3}{6} - x^2 + x - 1\right) dx = \frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1$$

Fourth approximation,

$$y^{(4)} = 1 + \int_{x_0}^{x} f(x, y^{(3)})\, dx = 1 + \int_0^x (x - y^{(3)})\, dx$$

$$= 1 + \int_0^x \left(x - \frac{x^4}{24} + \frac{x^3}{3} - x^2 + x - 1 \right) dx$$

$$= -\frac{x^5}{120} + \frac{x^4}{12} - \frac{x^3}{3} + x^2 - x + 1$$

Fifth approximation, $y^{(5)} = 1 + \int_{x_0}^{x} f(x, y^{(4)})\, dx = 1 + \int_0^x (x - y^{(4)})\, dx$

$$= 1 + \int_0^x \left(\frac{x^5}{120} - \frac{x^4}{12} + \frac{x^3}{3} - x^2 + 2x - 1 \right) dx$$

$$= \frac{x^6}{720} - \frac{x^5}{60} + \frac{x^4}{12} - \frac{x^3}{3} + x^2 - x + 1 \text{ etc.}$$

When $x = 0.2$, we have

$$y_0 = 1,\ y^{(1)} = 0.82,\ y^{(2)} = 0.83867,\ y^{(3)} = 0.83740.$$

$$y^{(4)} = 0.83746 \text{ and } y^{(5)} = 0.83746, \text{ i.e., } y = 0.837 \text{ at } x = 0.2.$$

Example 3: ***Use Picard's method to approximate the value of y when*** $x = 0.1$, ***given that*** $y = 1$ ***when*** $x = 0$ ***and*** $(dy/dx) = 3x + y^2$.

Solution: We have $f(x, y) = 3x + y^2$, $x_0 = 0$ and $y_0 = 1$.

Now first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 1 + \int_0^x (3x + y_0{}^2)\, dx$$

$$= 1 + \int_0^x (3x + 1)\, dx = \frac{3x^2}{2} + x + 1.$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x [3x + \{y^{(1)}\}^2]\, dx = 1 + \int_0^x \left(\frac{9}{4} x^4 + 3x^3 + 4x^2 + 5x + 1 \right) dx$$

$$= \frac{9}{20} x^5 + \frac{3}{4} x^4 + \frac{4}{3} x^3 + \frac{5}{2} x^2 + x + 1.$$

Third approximation, $y^{(3)} = 1 + \int_0^x [3x + \{y^{(2)}\}^2]\, dx$

$$= 1 + \int_0^x \left(\frac{81}{400} x^{10} + \frac{27}{40} x^9 + \frac{141}{80} x^8 + \frac{17}{4} x^7 \right.$$

$$\left. + \frac{1157}{180} x^6 + \frac{136}{240} x^5 + \frac{125}{12} x^4 + \frac{23}{3} x^3 + 6x^2 + 5x + 1 \right) dx$$

$$= \frac{81}{4400}x^{11} + \frac{27}{400}x^{10} + \frac{47}{240}x^{9} + \frac{17}{32}x^{8} + \frac{1157}{1260}x^{7} + \frac{62}{45}x^{6}$$
$$+ \frac{25}{12}x^{5} + \frac{23}{12}x^{4} + 2x^{3} + \frac{5}{2}x^{2} + x + 1$$

When $x = 0.1$, we get $y_0 = 1,\ y^{(1)} = 1.115,\ y^{(2)} = 1.1264,\ y^{(3)} = 1.12721;$

$\Rightarrow$ $y = 1.127$ at $x = 0.1$

Example 4: ***Use Picard's method to approximate y when $x = 0.1$ given that $y = 1$, when $x = 0$ and*** $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$.

[Bhopal 2002; Avadh 2005]

Solution: We have $f(x, y) = \dfrac{y-x}{y+x}, x_0 = 0, y_0 = 1.$

Now, first approximation,

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\,dx = 1 + \int_0^x \frac{1-x}{1+x}\,dx$$

$$y^{(1)} = 1 + \{2\log(1+x) - x\}_0^x = 1 - x + 2\log(1+x)$$

Second approximation, $y^{(2)} = x_0 + \displaystyle\int_{x_0}^x f(x, y^{(1)})\,dx = 1 + \int_0^x \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}\,dx$

$$= 1 + x - 2\int_0^x \frac{x}{1+2\log(1+x)}\,dx$$

$$= 1 + x - 2\int_0^x \frac{e^{2t}}{1+2t}\,dt + 2\int_0^t \frac{e^t}{1+2t}\,dt,$$

where $t = \log(1+x)$ which is difficult to integrate.

When $x = 0,\ y = 1.$

When $x = 0.1, y^{(1)} = 1 - 0.1 + 2\log(1.1) = 0.9828$

$y^{(2)} =$ can not be obtained.

Hence in this example, only first approximation, which is 0.9828, can be obtained.

Example 5: ***Use Picard's method to approximate y when $x = 0.1, x = 0.2$ given that $y = 1$ when $x = 0, \dfrac{dy}{dx} = x + y$. Check the result with the exact value.***

Solution: We have $(dy/dx) = f(x, y) = x + y, x_0 = 0, y_0 = 1.$

Now first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0)\,dx = 1 + \int_0^x (1+x)\,dx = 1 + x + \frac{x^2}{2}$$

Second approximation, $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx = 1 + \int_0^x \left\{x + \left(\frac{x^2}{2} + x + 1\right)\right\} dx$

$$= (x^3/6) + x^2 + x + 1.$$

Third approximation, $y^{(3)} = y_0 + \int_0^x \left\{x + \left(\frac{x^3}{6} + x^2 + x + 1\right)\right\} dx = \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1.$

When $x = 0.1$, we have $y^{(1)} = 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$

$$y^{(2)} = 1 + (0.1) + (0.1)^2 + \frac{1}{6}(0.1)^3 = 1.11016$$

$$y^{(3)} = 1 + 0.1 + (0.1)^2 + \frac{1}{3}(0.1)^3 + \frac{1}{24}(0.1)^4 = 1.1103$$

Closer approximation.
When $x = 0.2$, we further have

$$y^{(3)} = \frac{1}{24}(0.2)^4 + \frac{1}{3}(0.2)^3 + (0.2)^2 + 0.2 + 1 = 1.2427$$

We can continue further to get the better approximations.
Now, we shall obtain the exact value

$$\frac{dy}{dx} = x + y, \qquad \Rightarrow \qquad \frac{dy}{dx} - y = x$$

This is linear differential equation of the form $\frac{dy}{dx} + Py = Q$, whose I.F. $= e^{\int P\,dx}$ and solution is $y\,.\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

$\Rightarrow$ $$y e^{-x} = \int x e^{-x}\,dx + c = -e^{-x}(1 + x) + c$$

$\Rightarrow$ $$y = -x - 1 + c e^x.$$

Putting $y = 1, x = 0$, we obtain

$$1 = -1 + c \Rightarrow c = 2, \qquad \therefore \quad y = -x - 1 + 2e^x$$

When $x = 0.1,\ y = 2e^{0.1} - 0.1 - 1 = 1.11034.$

When $x = 0.2,\ y = 2e^{0.2} - 0.2 - 1 = 1.2428.$

These result reveal that the approximations obtained for $x = 0.1$ is correct to four decimal places while the approximation for $x = 0.2$ is correct to three decimal places.

Example 6: ***Approximate y and z by using Picard's method for the particular solution of***

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2.$$

Given that $y = 2,\ z = 1$, when $x = 0$.

[B.C.A. (Meerut) 2006, 2008; B.C.A. (Avadh) 2004]

Solution: Let $f(x, y, z) = x + z$ and $\phi(x, y, z) = x - y^2$.

$$x_0 = 0,\ y_0 = 2,\ z_0 = 1.$$

$\Rightarrow$ $$\frac{dy}{dx} = f(x, y, z) \qquad \Rightarrow \qquad y = y_0 + \int_{x_0}^{x} f(x, y, z)\,dx \qquad \ldots(1)$$

Again, we have $$\frac{dz}{dx} = \phi(x, y, z) \quad \Rightarrow \quad z = z_0 + \int_{x_0}^{x} \phi(x, y, z)\, dx \qquad \ldots(2)$$

Now, first approximation of y is

$$y^{(1)} = y_0 + \int_0^x f(x, y_0, z_0)\, dx = 2 + \int_0^x f(x, y_0, z_0) \qquad \text{[using (1)]}$$

$$= 2 + \int_0^x (x+1)\, dx = 2 + x + (x^2/2).$$

Also, first approximation of z is

$$z^{(1)} = z_0 + \int_0^x \phi(x, y_0, z_0)\, dx \qquad \text{[using (2)]}$$

$$= 1 + \int_0^x (x-4)\, dx = 1 - 4x + (x^2/2).$$

Hence the second approximation of $y^{(2)}$ and $z^{(2)}$ are

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)}, z^{(1)})\, dx$$

$$= 2 + \int_0^x (x + z^{(1)})\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{x^2}{2}\right) dx$$

$$= 2 + x - (3/2)\, x^2 + (1/6)\, x^3$$

and $$z^{(2)} = z_0 + \int_0^x \phi(x, y^{(1)}, z^{(1)})\, dx = 1 + \int_0^x [x - \{y^{(1)}\}^2]\, dx$$

$$= 1 + \int_0^x \left[x - \left(2 + x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 - 4x - \frac{3}{2}x^2 - \frac{2}{3}x^3 - (x^4/4) - (x^5/20).$$

9.2 Euler's Method

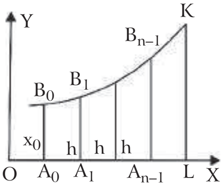
[Agra 2003, 2006]

Euler's method, as such, is of very little practical importance, but illustrates in simple form, the basic idea of those numerical methods which seek to determine the change Δy in y corresponding to small increase in the argument x.

Let the differential equation be

$$\frac{dy}{dx} = f(x, y), \quad \text{where } y = y_0 \text{ at } x = x_0. \qquad \ldots(1)$$

Again let $y = f(x)$ be the solution of (1) and let its graph be as shown in the adjoining diagram. We want to find out the value of y at $x = L$, say. For this divide the interval $l - x_0$ in 'n' equal parts. Let $x_1, x_2, \ldots\ldots, x_{n-1}$ be the intermediate points.



OL=1, OA$_0$=x$_0$

Fig. 9.1

Now, in the interval (x_0, x_1) we have an approximate relation.

$$y_1 - y_0 \approx (x_1 - x_0)\left(\frac{dy}{dx}\right)_{x=x_0} = (x_1 - x_0) f(x_0, y_0)$$

$$\therefore \qquad y_1 \approx y_0 + hf(x_0, y_0)$$

$$\tan\theta = f(x_0, y_0) = \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{MR}{B_0\,R}$$

$$\approx \frac{B_1\,R}{B_0\,R} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

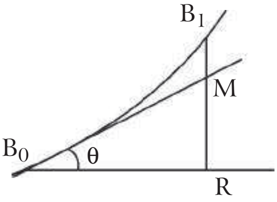


Fig. 9.2

$$\Rightarrow \qquad y_1 = y_0 + h(\Delta y / \Delta x)$$

$$\approx y_0 + hf\,(x_0, y_0)$$

This y_1 is approximate value of y for $x = x_1$.

On the same lines, the approximate value of y at $x = x_2$ is given by

$$y_2 \approx y_1 + hf(x_0 + h, y_1) = y_1 + hf(x_1, y_1) \qquad \ldots(2)$$

Repeating this procedure n times, we get

$$y_n\,(i.e.,\ LK) \approx y_{n-1} + hf\,(x_0 + (n-1)\,h, y_{n-1}) = y_{n-1} + hf\,(x_{n-1}, y_{n-1})$$

This is **Euler's Method.**

Note 1: A great disadvantage of the above method lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, then its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from this may be in much error from its true value. Errors of this type will be accumulated in the succeeding intervals. Finally, value of y becomes so much in error that it is completely useless.

Note 2: In Euler's method, the actual solution curve is approximated by the sequence of short straight lines, and sometimes the sequence of short lines diviates from the solution curve significantly. To avoid this difficulty. Euler's improved method is preferred as there we consider the curvature of the actual curve instead of approximating the curve by the sequence of short lines.

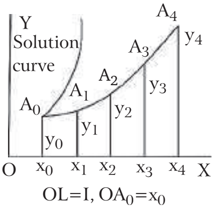


Fig. 9.3

Example 7: ***Solve the equation*** $\frac{dy}{dx} = 1 - y$ ***with the initial condition*** $x = 0, y = 0$ ***using Euler's method and tabulate the solutions at*** $x = 0.1, 0.2, 0.3.$

[B.C.A. (Meerut) 2005, 2007, 2008, 2012; B.C.A. (Purvanchal) 2006, 2010; B.C.A. (Bundelkhand) 2008]

Solution: We have $f(x, y) = 1 - y$

Let $\qquad h = 0.1,\ x_0 = 0,\ y_0 = 0,$ we find

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 0 + (0.1)(1 - 0) = 0.1$$

$\therefore \quad y(0.1) = 0.1$

Also $\quad y_2 = y_1 + hf(x_1, y_1)$

$$= 0.1 + (0.1)(1 - 0.1) = 0.1 + 0.09 = 0.19$$

$\therefore \quad y(0.2) = 0.19$

Again $\quad y_3 = y_2 + hf(x_2, y_2)$

$$= 0.19 + (0.1)(1 - 0.19)$$

$$= 0.271$$

$\therefore \quad y(0.3) = 0.271$

The tabulated values are

x	$y(x)$
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example 8: ***(i) Given*** $\frac{dy}{dx} = \frac{y - x}{y + x}$ ***with the initial condition*** $y = 1$ ***at*** $x = 0$***. Find*** y ***for*** $x = 0.1$***.*** **[Agra 2003; Avadh 2003]**

(ii) Using Euler's method, find an approximate value of y ***corresponding to*** $x = 1$***, given that*** $(dy/dx) = x + y$ ***and*** $y = 1$ ***where*** $x = 0$***.***

Solution: (i) We have

$$\frac{dy}{dx} = \frac{y - x}{y + x} = f(x, y), \ x_0 = 0, \ y_0 = 1, \ h = 0.1.$$

Hence the approximate value of y at $x = 0.1$ is given by

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1\left\{\frac{1-0}{1+0}\right\} = 1.1.$$

Much better accuracy is obtained by breaking up the interval 0 to 0.1 into five steps.

The approximate value of y at $x_A = 0.02$ is given by

$$y_A = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.02\,\frac{1-0}{1+0} = 1.0200,$$

$$\therefore \quad h = \frac{0.1}{5} = 0.02$$

The approximate value of y at $x_B = 0.04$ is given by

$$y_B = y_A + hf(x_A, y_A)$$

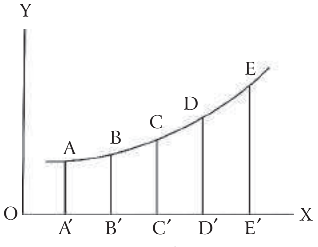


Fig. 9.4

$$=1.0200+.02\,\frac{1.02-0.02}{1.02+0.02} \qquad (\because\ x_A=0.02\,,\ y_A=1.02)$$

$$=1.0200+0.02\,\frac{1}{1.04}=1.02+0.0192=1.0392.$$

The approximate value of y at $x_C=0.06$ is given by

$$y_C=y_B+hf\,(x_B,y_B)=1.0392+0.02\,\frac{1.0392-0.04}{1.0392+0.04}$$

$$[\therefore\ x_B=0.06,\ y_B=1.0392]$$

$$=1.0392+0.02\,\frac{0.9992}{1.0792}-1.0392+0.02\,(0.926)$$

$$=1.0392+0.0185=1.0577.$$

The approximate value of y at $x_D=0.008$ is given by

$$y_D=y_C+hf\,(x_C,y_C)=1.0571+h\,\frac{h_C-x_C}{y_C+x_C}$$

$$=1.0577+0.02\,\frac{(0.9377)}{(1.1177)}$$

$$=1.0577+0.02\,(0.893)=1.0577+0.0179=1.0756.$$

The approximate value of y at $x_E=0.1$ is given by

$$y_E=y_D+hf\,(x_D,y_D)=y_D+h\,\frac{y_D-x_D}{y_D+x_D}$$

$$=1.0756+0.02\,\frac{1.0756-0.08}{1.0756+0.08}=1.0756+0.02\,\frac{0.9956}{1.1556}$$

$$=1.0756+0.02\,(0.862)=1.0756+0.172=1.0928.$$

(ii) Let us take $n=10$; $h=0.1$ (sufficiently small). Then various calculations are arranged as follows:

x	y	$x+y=(dy/dx)$	*Old* $y+0.1\,(dy/dx)=$ *new* y
0.0	1.00	1.00	1.00 + 0.1 (1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1 (1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1 (1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1 (1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1 (1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1 (2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1 (2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1 (2.89) = 2.48
0.8	2.48	3.29	2.48 + 0.1 (3.29) = 2.81
0.9	2.81	3.71	2.81 + 0.1 (3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y=3.18$.

9.3 Improved Euler's Method

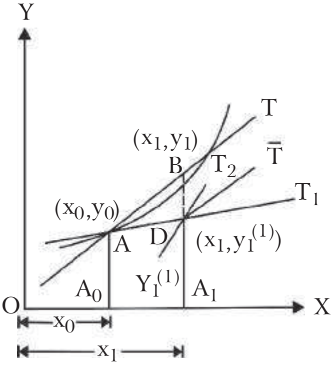


Fig. 9.5

In this method, we consider a line passing through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ such that $y_1^{(1)} = y_0 + hf(x_0, y_0)$. Geometrically, AT_1 is the tangent at (x_0, y_0). DT_2 is the line through $(x_1, y_1^{(1)})$ having slope $(x_1, y_1^{(1)})$ and $D\bar{T}$ is the line having slope

$$\frac{f(x_0\,, y_0) + f(x_1, y_1^{(1)})}{2}$$

The line AT through (x_0, y_0) and parallel to $D\bar{T}$ is used to approximate the curve *i.e.*, the ordinate of point B will give the value of y_1. Obviously, the equation of the line ABT is given by

$$y - y_0 = (x - x_0)\left\{\frac{f\,(x_0, y_0) + f\,(x_1, y_1^{(1)})}{2}\right\} \qquad \ldots(1)$$

As we are assuming $A_1\,B = y_1$, so co-ordinates of B will be (x_1, y_1). This point lies on the line AT, so we have

$$y_1 - y_0 = (x_1 - x_0)\left\{\frac{f\,(x_0, y_0) + f\,(x_1, y_1^{(1)})}{2}\right\} \qquad \ldots(2)$$

Continuing further, we get

$$y_{m-1} - y_m = (x_{m+1} - x_m)\left\{\frac{f\,(x_m, y_m) + f\,(x_{m+1}, y_{m+1}^{(1)})}{2}\right\}$$

or $$y_{m-1} = y_m + \frac{h}{2}\{f\,(x_m, y_m) + f\,(x_{m+1}\,, y_{m+1}^{(1)})\}$$

where $$x_{m+1} - x_m = h \qquad \ldots(3)$$

which agrees with the term obtained by the integration of $y' = f(x)$ using trapezoidal rule.

9.4 Modified Euler's Method

[Bangaluru 2004, 2007]

Starting with the initial value y_0 an approximate value for y_1 is computed from the relation

$$y_1^{(1)} \approx y_0 + h\left(\frac{dy}{dx}\right)_{x=x_0} = y_0 + f\,(x_0, y_0)\,h \qquad \ldots(1)$$

where $(dy/dx) = f(x, y)$ and $y_1^{(1)}$ is the first approximation of y at $x = x_1$.

After obtaining the value of $y_1^{(1)}$, we substitute it in the differential equation.

$\left(\frac{dy}{dx}\right) = f(x, y)$, so as to get the approximate value of $\left(\frac{dy}{dx}\right)$ at the end of first interval *viz.*,

$$\left(\frac{dy}{dx}\right)_1^{(1)} = f(x_1, y_1^{(1)}) \qquad \ldots(2)$$

The improved value of $\Delta\, y$ is then obtained as given below:

$$\Delta\, y \approx [\text{Mean value of } \frac{dy}{dx} \text{ at the end of the interval } x_0 \text{ to } x_1]\, h$$

$$= \frac{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(1)}}{2}\, h. \qquad \ldots(3)$$

Hence, the second approximation for y is given by

$$y_1^{(2)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(1)}}{2}\, h = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})] \quad \ldots(4)$$

The improved value $y_1^{(2)}$ is now substituted in the equation $\frac{dy}{dx} = f(x, y)$, so as to get second approximation for $\left(\frac{dy}{dx}\right)_1$ *viz.*,

$$\left(\frac{dy}{dx}\right)_1^{(2)} = f(x_1, y_1^{(2)}) \qquad \ldots(5)$$

The third approximation for y is therefore given by

$$y_1^{(3)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(2)}}{2}\, h = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(2)})] \quad \ldots(6)$$

The process is stopped, when the value of $y_1^{(n-1)}$ and $y_1^{(n)}$ are essentially the same

$$y_1^{(n)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}^{(n-1)}}{2}\, h = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(n-1)})] \quad \ldots(7)$$

First approximation to $y_2, y_3, \ldots$ etc., could be found by means of formula.

$y_{n+1} \approx y_n + h\left(\frac{dy}{dx}\right)_n$, but as soon as two consecutive values of y are known, the first approximation to succeeding y's can be obtained more accurately by the formula

$$y_{n+1} = y_{n-1} + 2h\left(\frac{dy}{dx}\right)_n = y_{n-1} + 2h\, y'_n \qquad \ldots(8)$$

Proof of (8) is not given here but is obtained by making use to Taylor's series in the neighbourhood of x_n.

By repeating this process, we can obtain the approximate value of y at $x = 1$.

Geometrical Illustration: *For the improvement of this method over Euler's method.*

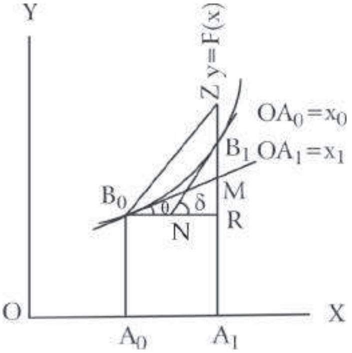


Fig. 9.6

Δy by Euler's method as computed is represented by $R\,M$. If $B_0\,N$ is drawn parallel to the tangent at B_1, then $R\,N$ denotes the computed value of Δy by using the slope at B_1. But if we take the average value of the slopes, then the computed value of $(\Delta y / \Delta x)$ becomes

$$\frac{\Delta y}{\Delta x} = \frac{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}}{2}$$

$$\Delta y = \frac{h}{2}\left\{\left(\frac{dy}{dx}\right)_{x_0} + \left(\frac{dy}{dx}\right)_{x_1}\right\}$$

where $\quad h = x_1 - x_0$

$$= \frac{1}{2}\left\{h\left(\frac{dy}{dx}\right)_{x_0} + h\left(\frac{dy}{dx}\right)_{x_1}\right\}$$

$$= \frac{1}{2}[h \tan\theta + h \tan\delta]$$

$$= \frac{1}{2}[B_0\,R \tan\theta + B_0\,R \tan N\,B_0\,R]$$

$$= \frac{1}{2}[RM + RN]$$

Hence $\quad \Delta y = \frac{1}{2}[RM + RM + MN] = RM + \frac{MN}{2}$

which is in close approximation with its true value $R\,B_1$.

Example 9: ***Given:*** $\frac{dy}{dx} = x - y^2$, $y(0.02) = 0.02$ ***Find*** $y(0.4)$ ***by modified Euler's method correct to*** **3** ***decimal places, taking*** $h = 0.2$.

[B.C.A. (Meerut) 2011; B.C.A. (Gorakhpur) 2008, 2011]

Solution: We have $f(x, y) = x - y^2$; $x_0 = 0.2$, $y_0 = 0.02$ and $h = 0.2$

Let $\quad x_1 = 0.4$ then we have to find $y_1 = y(0.4)$

Now $\quad f(x_0, y_0) = x_0 - y_0^2 = 0.2 - (0.02)^2$

$$= 0.2 - 0.0004 = 0.1996$$

$\therefore \quad y_1^{(1)} = y_0 + hf(x_0, y_0) = 0.02 + (0.2)(0.1996) = 0.060$

Again $$f(x_1, y_1^{(1)}) = x_1 - \left[y_1^{(1)} \right]^2 = 0.4 - (0.06)^2 = 0.3964$$

$\therefore$ $$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 0.02 + \frac{0.2}{2}[0.1996 + 0.3964] = 0.0796 \approx 0.080$$

Again $$f(x_1, y_1^{(2)}) = x_1 - [y_1^{(2)}]^2 = 0.4 - (0.08)^2 = 0.3936$$

$\therefore$ $$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 0.02 + \frac{0.2}{2}[0.1996 + 0.3936] = 0.07932 \approx 0.079$$

Again $$f(x_1, y_1^{(3)}) = x_1 - \{y_1^{(3)}\}^2 = 0.4 - (0.079)^2 = 0.3938$$

$\therefore$ $$y_1^{(4)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(3)})\}$$

$$= 0.02 + \frac{0.2}{2}\{0.1996 + 0.3938\}$$

$$= 0.0793 \approx 0.079$$

Since $y_1^{(3)} = y_1^{(4)}$, therefore $y_1(0.4) = 0.079$

Example 10:* *Given that $\frac{dy}{dx} = x + y$, with the initial condition that $y = 1$, when $x = 0$. Find y for $x = 0.05$, and $x = 0.1$. Give the correct result upto four decimal places.

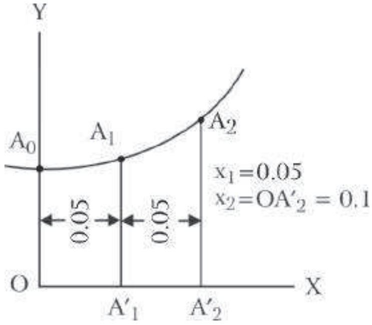


Fig. 9.7

Solution: The given differential equation is

$$\frac{dy}{dx} = x + y = f(x, y), \text{ say.}$$

Initial condition is $x_0 = 0, y_0 = 1.$

This gives $$\left(\frac{dy}{dx}\right)_0 = x_0 + y_0 = 0 + 1 = 1.$$

Now, $$y_1^{(0)} = y_0 + f(x_0, y_0) \times h = 1 + (0+1)(0.05) = 1.05,$$

$$\left(\frac{dy}{dx}\right)_0^{(1)} = f(x_1, y_1^{(1)}) = (x_1 + y_1^{(1)}) = 0.05 + 1.05 = 1.10$$

$$y_1^{(2)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2} h = 1 + \frac{1 + 1.10}{2}(0.05) = 1.0525,$$

The second approximation to $\left(\frac{dy}{dx}\right)$, is

$$\left(\frac{dy}{dx}\right)_1^{(2)} = f(x_1, y_1^{(2)}) = (x_1 + y_1^{(2)}) = 0.05 + 1.0525 = 1.1025,$$

Now the third approximation to y_1 is

$$y_1^{(3)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(2)}}{2} h = 1 + \frac{1 + 1.1025}{2}(0.05) = 1.05256,$$

The third approximation to $\left(\frac{dy}{dx}\right)$, is

$$\left(\frac{dy}{dx}\right)_1^{(3)} = f(x_1, y_1^{(3)}) = x_1 + y_1^{(3)} = 0.05 + 1.05256 = 1.10256,$$

The fourth approximation to y_1 is

$$y_1^{(4)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(3)}}{2} h = 1 + \frac{1 + 1.10256}{2}(0.05) = 1.05256.$$

It is obvious that $y_1^{(3)} = y_1^{(4)}$, so we stop here and say that

$$y = 1.05256, \left(\frac{dy}{dx}\right)_1 = 1.1026$$

As a first approximation to y_2, we have

$$y_2^{(1)} = y_0 + 2.h\left(\frac{dy}{dx}\right)_1 = 1 + 2(0.05)(1.1026) = 1.1103$$

[Using equation (8) of article **9.4**]

Now $$y_2^{(2)} = y_1 + h\frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(1)}}{2}$$

$$= 1.0526 + (0.05)\frac{1.1026 + (0.1 + 1.1103)}{2} = 1.1104.$$

and $$\left(\frac{dy}{dx}\right)_2^{(2)} = 0.1 + 1.1104 = 1.2104$$

Then $$y_2^{(3)} = y_1 + h\frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(2)}}{2}$$

$$= 1.0526 + (0.05)\frac{1.1026 + 1.1104)}{2} = 1.1104$$

Hence $$y_2^{(2)} = y_2^{(3)} = 1.1104.$$

This shows that the value of y at $x = 0.1$, is $y = 1.1104$

Also $$\left(\frac{dy}{dx}\right)_2 = 0.1 + 1.1104 = 1.2104$$

Collecting results in the tabulated form, we have the following table:

x	y	(dy / dx)
0.000	1.0000	1.0000
0.05	1.052	1.1026
0.10	1.1104	1.2104

Example 11: ***Given that*** $\frac{dy}{dx} = \log(x+y)$, ***with the initial condition that*** $y = 1$ ***when*** $x = 0$, ***find*** y ***for*** $x = 0.2$ ***and*** $x = 0.5$ ***by using Euler's modified method.*** **[Agra 2000]**

Solution: Let $x = 0$, $x_1 = 0.2$ and $x_2 = 0.5$ then $y_0 = 1$, while y_1 and y_2 are to be computed.

Here $\quad f(x, y) = \log(x + y), \qquad \therefore \quad f(x_0, y_0) = \log(1) = 0$

Now, $\quad y_1^{(1)} = y_0 + hf(x_0, y_0) = 1 + (0.2)(0) = 1 \qquad (\because \quad h = 0.2)$

$$\left(\frac{dy}{dx}\right)_1^{(1)} = f(x_1, y_1^{(1)}) = \log(x_1 + y_1^{(1)}) = \log(0.2 + 1) = \log(1.2)$$

$$\therefore \quad y_1^{(2)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2} h = 1 + \frac{f(x_0, y_0) + \log(1.2)}{2} \times (0.2)$$

$$= 1 + \frac{0 + \log(1.2)}{2}(0.2) = 1 + 0.0079 = 1.0079,$$

Now, $$\left(\frac{dy}{dx}\right)_1^{(2)} = f(x_1, y_1^{(2)}) = \log(x_1 + y_1^{(2)}) = \log[0.2 + 1.0079],$$

$$y_1^{(3)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(2)}}{2} h = 1 + \frac{0 + \log(0.2 + 1.0079)}{2} \times (0.2)$$

$$= 1 + 0.0082 = 1.0082,$$

$$\left(\frac{dy}{dx}\right)_1^{(3)} = f(x_1, y_1^{(3)}) = \log(x_1 + y_1^{(3)}) = \log(0.2 + 1.0082)$$

$$y_1^{(4)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(3)}}{2} h = 1 + \frac{0 + \log(0.2 + 1.0082)}{2} \times (0.2)$$

$$= 1 + 0.0082 = 1.0082.$$

It is obvious that $y_1^{(3)} = y_1^{(4)}$, so we stop here and say that $y_1 = 1.0082$.

To obtain y_2, the value of y at $x = 0.5$, we again make use of equation

$$y_2^{(1)} = y_1 + hf(x_1, y_1), \text{ giving that}$$

$$y_2^{(1)} = 1.0082 + 0.3 \log(0.2 + 1.0082) = 1.0082 + 0.0246 = 1.0328.$$

$(\because \; h = 0.5 - 0.2 = 0.3 \text{ here})$

To improve this estimate of y_2, we have

$$y_2^{(2)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(1)}}{2} h$$

$$= y_1 + \frac{\{\log (x_1 + 1.0012) + \log (x_2 + y_2^{(1)})\}}{2}(0.3)$$

$$= 1.0082 + \frac{\{\log (1.2 + 1.0082) + \log (0.5 + 1.0328)\}}{2}(0.3)$$

$$= 1.0082 + 0.0401 = 1.0483.$$

Now $$y_2^{(3)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(2)}}{2} h$$

$$= 1.0082 + \frac{\{\log (0.2 + 1.0082) + \log (0.5 + 1.0483)\}}{2} \times (0.3)$$

$$= 1.0082 + 0.0408 = 1.0490.$$

Another time, the value will not change and hence it is taken as the value of y at $x = 0.5$.

Problem Set-1

1. Using Euler's method, find approximate value of y when $x = 0.6$ of $(dy/dx) = 1 - 2xy$, given that $y = 0$ when $x = 0$ (take $h = 0.2$).
2. Using simple Euler's method solve for y at $x = 0.1$ from $(dy/dx) = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
3. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length $= 0.2$ (carry out 6 step).

Answers-1

1.	0.4748	2.	1.1448
3.	1.1831808		

9.5 Runge-Kutta Method

[B.C.A. (Agra) 2003, 2005, 2008; B.C.A. (Lucknow) 2005, 2008, 2010]

Actually, Taylor's series method of solving differential equations numerically involves lot of labour in finding out the higher order derivatives. ***R-K method*** then was introduced and does not require the calculation of higher order derivatives and also gives greater accuracy. ***R-K method*** only requires the functional values at some selected points and agree with Taylor's series solution up to term in h^r where r differs from method to method and is known as the order of that method.

9.5.1 R-K Method of First Order

[Agra 2007]

By Euler's method, we have $y_1^{(1)} = y_0 + hf(x_0, y_0) = y_0 + h\,y'_0$ where $(dy/dx) = f(x, y)$

Also expanding by Taylor's series method, we have

$$\begin{aligned} y_1^{(1)} &= y_1 \text{ say} = y_0\,(x+h) \\ &= y_0 + h\,y'_0 + (h^2/2\,!)\,y''_0 + (h^2/3\,!)\,y'''_0 + \ldots. \end{aligned}$$

This means that Euler's method agrees with Taylor's series solution up to the term in h. *Hence Euler's method is nothing but R-K method of the first order.*

9.5.2 R-K Method of Second Order

By modified Euler's method, we have

$$y_1 = y_0 + \frac{h}{2}\,[f\,(x_0, y) + f\,(x_0 + h, y_1)] \qquad \ldots(1)$$

Putting $y_1 = y_0 + hf\,(x_0, y_0)$ in the **R.H.S.**, we get

$$y_1 = y_0 + \frac{h}{2}\,[f_0 + f\,(x_0 + h, y_0 + h\,f_0)] \text{ where } f(x_0, y_0) = f_0 \qquad \ldots(2)$$

Again
$$\begin{aligned} y_1 &= y\,(x_0 + h) \\ &= y_0 + h\,y'_0 + (h^2/2\,!)\,y''_0 + (h^3/3\,!)\,y'''_0 + \ldots. \end{aligned}$$

By Taylor's series and

$$f\,(x_0 + h, y_0 + h\,f_0) = f\,(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + h\,f_0\left(\frac{\partial f}{\partial y}\right)_0 + \text{terms having } h^2, h^3 \text{ etc.}$$

∴ Relation (2)

$$\Rightarrow \qquad y_1 = y_0 + \frac{h}{2}\left[f_0 + f_0 + h\left(\frac{\partial f}{\partial x}\right)_0 + h\,f_0\left(\frac{\partial f}{\partial y}\right)_0 + O\,(h)^3\right]$$

$$= y_0 + h\,f_0 + (h^2/2\,!)\left[\left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0\right] + \ldots$$

$$\Rightarrow \qquad y_1 = y_0 + h\,f_0 + (h^2/2\,!)\left[\frac{d}{dx}\,f(x, y)\right]_0 + \ldots \qquad \left[\because\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{du}{dx}\right]$$

$$= y_0 + h\,y'_0 + (h^2/2\,!)\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] + \ldots$$

$$= y_0 + hy'_0 + (h^2/2\,!)\,y''_0 + O\,(h^3) \qquad \ldots(3)$$

Also
$$\begin{aligned} y_1 &= y_0\,(x+h) \\ &= y_0 + hy'_0 + (h^2/2\,!)\,y''_0 + O\,(h^3) \qquad \ldots(4) \end{aligned}$$

From (3) and (4), we see that the modified Euler's method is in agreement with Taylor's series solution up to the term in h^2. Thus, we can say that the modified Euler's method is nothing but ***R-K method*** of second order.

Working rule

R-K method of second order is $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = hf(x_0, y_0), k_2 = hf(x_0 + h, y_0 + k_1)$

9.5.3 R-K Method of Third Order

To find the increment k of y corresponding to an increment h of x by ***R-K method*** from $(dy/dx) = f(x, y)$ where $y(x_0) = y_0$, we proceed as follows:

Calculate $k_1 = hf(x_0, y_0), k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right],$

$$k_3 = hf[x_0 + h, y_0 + k'];$$

where $k' = hf[x_0 + h, y_0 + k_1]$ Finally compute

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

9.5.4 R-K Method of Fourth Order (or Simply Runge-Kutta Method)

To find the increment k of y corresponding to an increment h in x by ***R-K method*** from $(dy/dx) = f(x, y), y(x_0) = y_0$; we calculate:

$$k_1 = hf(x_0, y_0), k_2 = hf\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right]$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf[x_0 + h, y_0 + k_3]$$

and finally $k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$; so that $y_1 = y_0 + k$.

Example 12: ***Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $(dy/dx) = x + y$ and $y = 1$ when $x = 0$.***

[Agra 2004; Meerut 2002]

Solution: $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

Hence, we get $k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$

$\Rightarrow$ $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$

$$= \frac{1}{6} \times 1.4568 = 0.2428$$

Hence the required approximate value of $y = y_0 + 0.2428 = 1.2428$

Example 13: ***Apply Runge-Kutta method to find approximate value of y for $x = 0.2$, in steps of 0.1, if $(dy/dx) = x + y^2$, given that $y = 1$ where $x = 0$. Given $f(x, y) = x + y^2$.***

[Agra 2003]

Solution: Let us choose $h = 0.1$ and calculate various values as follows:

Step 1: $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\Rightarrow$ $k_1 = hf(x_0, y_0) = 0.1\, f(0,1) = 0.1000$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = 0.1\, f(0.05, 1.1) = 0.1152$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = 0.1 \times f(0.05, 1.1152) = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 1.1168) = 0.1347$$

and $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) = \mathbf{0.1165}$$

So that $y(0.1) = y_0 + k = 1.1165$

Step 2: $x_1 = x_0 + h = 0.1,\ y_1 = 1.1165,\ h = 0.1$

$\Rightarrow$ $k_1 = hf(x_1, y_1) = 0.1\, f(0.1, 1.1165) = 0.1347$

$$k_2 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) = 0.1\, f(0.15, 1.1838) = 0.1551$$

$$k_3 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = 0.1 f(0.15, 1.194) = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1 f(0.2, 1.1576) = 0.1823$$

and $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$

So that $y(0.2) = y_1 + k = 1.2736$.

Example 14: ***Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.***

Solution: $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2} = \dfrac{dy}{dx}$

To find $y(0.2)$:

We have $x_0 = 0,\ y_0 = 1,\ h = 0.2$

$\Rightarrow$ $k_1 = hf(x_0, y_0) = 0.2\ f(0, 1) = 0.2000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2\ f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2\ f(0.1,\ 1.09836) = 0.1967$$

$$k_4 = hf\,(x_0 + h, y_0 + k_3) = 0.2\ f(0.2,\ 1.1967) = 0.1891$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2\,(0.19672) + 2\,(0.1967) + 0.1891] = 0.19599$$

So that $y(0.2 = y_0 + k = 1.196)$.

To find $y(0.4)$:

We have $x_1 = 0.2,\ y_1 = 1.196,\ h = 0.2.$

⇒ $$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\ f(0.3,\ 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\ f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf\,(x_1 + h, y_1 + k_3) = 0.2 f(0.4,\ 1.3753) = 0.1688$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2\,(0.1795) + 2\,(0.1793) + 0.1688] = 0.1792$$

So that $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Problem Set-2

1. Using Runge-Kutta method of order 4, find $y(0.2)$ given that
$$(dy/dx) = 3x + \frac{1}{2}y,\ y(0) = 1,\ \text{taking } h = 0.1.$$

2. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $(dy/dx) = \sqrt{(x + y)}$.

3. Given that $(dy/dx) = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ as $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5.

4. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation
$$\frac{dy}{dx} = \frac{y - x}{y + x},\ y(0) = 1.\ \text{Take } h = 0.2.$$

5. Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from
$$10\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1,\ \text{taking } h = 0.1.$$

[B.C.A. (Agra) 2006; B.C.A. (Meerut) 2003, 2008, 2012]

Answers-2

1.	2.5005
2.	1.1678
3.	1.0911, 1.1677, 1.2352, 1.2902, 1.338.
4.	1.1749
5.	1.0207, 1.038

Example 15: ***Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $(dy/dx) = xy + y^2$, $y(0) = 1$.***

Solution: We have $f(x, y) = xy + y^2 = (dy/dx)$.

Step 1: To find $y(0.1)$:

We have $x_0 = 0,\ y_0 = 1, h = 0.1.$

$\Rightarrow$ $k_1 = hf(x_0, y_0) = (0.1) f(0, 1) = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 1.05) = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 1.0577) = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 1.1172) = 0.13598$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) = 0.11087$$

$\Rightarrow$ $y(0.1) = y_1 = y_0 + k = 1.1169.$

Step 2: To find $y(0.2)$:

We have $x_1 = 0.1,\ y_1 = 1.1169,\ h = 0.1,$

$\Rightarrow$ $k_1 = hf(x_1, y_1) = (0.1) f(0.1, 1.1169) = 0.1359$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 1.1848) = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 1.1959) = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 1.2778) = 0.1888$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

$$\Rightarrow \quad y(0.2) = y_2 = y_1 + k = 1.2773.$$

Step 3: To find $y(0.3)$:

$$x_2 = 0.2,\ y_2 = 1.2773,\ h = 0.1.$$

$$\Rightarrow \quad k_1 = hf(x_2, y_2) = (0.1)\ f(0.2, 1.2773) = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{h}, y_2 + \frac{1}{2}k_1\right) = (0.1)\ f(0.25, 1.3716) = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\ f(0.25, 1.3885) = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\ f(0.3,\ 1.5048) = 0.2716$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$$

$$\Rightarrow \quad y(0.3) = y_3 = y_2 + k = 1.504$$

Example 16:* *Solve the initial values problem $(dy/dx) = x - y^2$, $y(0) = 1$ to find $y(0.3)$ by using Runge-Kutta method of order 4 using step value $h = 0.1$.

Solution: $f(x, y) = x - y^2$.

Step 1: To find $y(0.1)$:

$x_0 = 0$, $y_0 = 1$, $h = 0.1$. Let us now compute

$$k_1 = hf(x_0, y_0) = (0.1)\ f(0, 1) = -0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\ f(0.05, 0.95) = -0.08525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\ f(0.05, 0.9574) = -0.0867$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\ f(0.1,\ 0.9137) = -0.07341$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0833$$

$$\therefore \quad y(0.1) = y_1 = y_0 + k = 1 - 0.0833 \quad = 0.9117$$

Step 2: To find $y(0.2)$:

$$x_1 = 0.1,\ y_1 = 0.9117,\ h = 0.1$$

$$\Rightarrow \quad k_1 = hf(x_1, y_1) = (0.1)\ f(0.1)\ f(0.1, 0.9117) = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\, f(0.15, 0.8751) = -0.0616$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\, f(0.15, 0.8809) = -0.0626$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\, f(0.2, 0.8491) = -0.0521$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0623$$

$\Rightarrow$ $$y(0.2) = y_2 = y_1 + k = 0.8494.$$

Step 3: To find $y(0.3)$:

$$x_2 = 0.2,\ y_2 = 0.8494,\ h = 0.1.$$

$\Rightarrow$ $$k_1 = hf(x_2, y_2) = (0.1)\, f(0.2, 0.8494) = -0.0521$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\, f(0.25, 0.8233) = -0.0428$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\, f(0.25, 0.828) = -0.0436$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\, f(0.3, 0.8058) = -0.0349$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0438$$

$\therefore$ $$y(0.3) = y_3 = y_2 + k = 0.8061$$

9.6 Simultaneous Differential Equation (First Order)

Simultaneous differential equation of the type

$$\frac{dy}{dx} = f(x, y, z) \qquad \ldots(1)$$

and $$\frac{dz}{dx} = \phi(x, y, z) \qquad \ldots(2)$$

With initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially Picard's or Runge-Kutta method.

9.6.1 Picard's Method

We have $$y_1 = y_0 + \int f(x, y_0, z_0)\, dx,\quad z_1 = z_0 + \int \phi(x, y_0, z_0)\, dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\, dx,\quad z_2 = z_0 + \int \phi(x, y_1, z_1)\, dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\, dx,\quad z_3 = z_0 + \int \phi(x, y_2, z_2)\, dx;\ \text{etc.}$$

9.6.2 Runge-Kutta Method

Starting initially (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method is

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$\therefore$ $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To calculate y_2 and z_2 we simply replace x_0, y_0, z_0, by x_1, y_1, z_1 in the above formulae and proceed.

Example 17: ***Using Picard's method, find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $(dy/dx) = x + z$, $(dz/dx) = x - y^2$.***

Solution: $x_0 = 0,\ y_0 = 2,\ z_0 = 1$. Now $\frac{dy}{dx} = f(x, y, z) = x + z$

$$\frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$\Rightarrow$ $$y = y_0 + \int_{x_0}^{x} f(x, y, z)\,dx$$

and $$z = z_0 + \int_{x_0}^{x} \phi(x, y, z)\,dx.$$

First Approximations $$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\,dx = 2 + \int_0^x (x+1)\,dx = 2 + x + \frac{1}{2}x^2$$

and $$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\,dx = 1 + \int_0^x (x-4)\,dx = 1 - 4x + \frac{1}{2}x^2.$$

Second Approximations $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx$

$$= 2 + \int_0^x (x + 1 - 4x + \frac{1}{2}x^2)\, dx = 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

and $z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx = 1 + \int_0^x [x - (2 + x + \frac{1}{2}x^2)^2]\, dx = 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$.

Third Approximations $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\, dx$

$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

and $z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\, dx = 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$ etc.

Thus at $x = 0.1$, we have $y_1 = 2.105$, $y_2 = 2.08517$, $y_3 = 2.08447$

$$z_1 = 0.605, z_2 = 0.58397, z_3 = 0.58672$$

Thus the correct to 4 decimal places, we have $y(0.1) = 2.0845$, $z(0.1) = 0.5867$.

9.7 Differential Equation of Second Order

Consider a second order differential equation as

$$\frac{d^2x}{dt^2} = \phi[x, (dx/dt), t), t]$$

Under the initial conditions $x(t_0) = x_0$ and $x'(t_0) = x'_0$. This second order equation can be reduced to a system of simultaneous differential equations of first order as follows:

Take $\frac{dx}{dt}$ to be y *i.e.*, $\frac{dx}{dt} = y$ or $x' = y$ ...(1)

and $\frac{dx}{dt} = \phi[x, y, t]$ ...(2)

Initial conditions being $(x)_{t_0} = x_0$ and $(y)_{t_0} = x'_0$.

It is worth noticing that an *n*th order differential equation yields *n* simultaneous differential equations of first order.

Example 18: ***Using Runge-Kutta method, solve*** $y'' = xy'^2 - y^2$ ***for*** $x = 0.2$ ***correct to 4 decimal places. Initial conditions are*** $x = 0$, $y = 1$, $y' = 0$.

Solution: Let $(dy/dx) = z = f(x, y, z)$. Then $\frac{dz}{dx} = xz^2 - y^2 = \phi(x, y, z)$

But $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, $h = 0.2$.

Then Runge-Kutta formulae become:

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0 \qquad l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.2(-0.1) = -0.02 \qquad = 0.2(-0.999) = -0.1998$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= 0.2(-0.0999) = -0.02 \qquad = 0.2(-0.9791) = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.2(-0.1958) = -0.0392 \qquad = 0.2(0.9527) = -0.1905$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= -0.0199 \qquad = -0.1970$$

Thus at $x = 0.2$, we have

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

Problem Set-3

1. Using Runge-Kutta method of order 4, solve $y'' = y + xy'$, $y(0) = 1, y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.

2. Use Euler's method to find $y(1)$ from the differential equation $\frac{dy}{dx} = \frac{-y}{1+x}, y(3) = 2$. Take for each step $h = 0.1$.

3. Use Picard's method to solve $\frac{dy}{dx} = 1 + xy$, with $x_0 = 2, y_0 = 0$.

4. Use Runge-Kutta method to solve $y' = xy$ for $x = 1.4$. Initially $x = 1, y = 2$ (take $h = 0.2$). **[Bhopal 2002; Agra 2006]**

5. (i) Integrate for $x = 0.1, 0.2$ the following initial value problem by Runge-Kutta's fourth order method.

$$y' = x - y^2, \quad y(0) = 1$$

   (ii) Use Runge-Kutta method to calculate the value of y at $x = 0.1$, to five places of decimal, after a single step of 0.1 if

$$y' = 0.31 + 0.25\,y + 0.3\,x^2$$

   and $\qquad y = 0.72 \quad \text{when } x = 0.$

6. Using Runge-Kutta's method with third order accuracy, solve $\frac{dy}{dx} = y - x$ with initial condition $y = 2, x = 0$.

7. Solve by Runge-Kutta's method $\frac{dy}{dx} = x + y$ with initial conditions $x_0 = 0$, $y_0 = 1$ correct upto four decimal places, by evaluating upto second increment of y (Take $h = 0.1$).

8. Solve the system of differential equations $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ for $x = 0.3$ given that $y = 0$ and $z = 1$ when $x = 0$, using Runge-Kutta's method of order four.

9. By the Fourth order Runge-Kutta's method, tabulate the solution of the differential equation

$$\frac{dy}{dx} = \frac{xy + 1}{10y^2 + 4}, \; y(0) = 0$$

   in $[0; 0.4]$ with step length 0.1 correct to five decimal places.

10. Using Fourth order Classical Runge-Kutta's method for the initial value problem $\frac{du}{dt} = -2tu^2, u(0) = 1$, with $h = 0.2$ on the interval $[0, 1]$, calculate $u(0.4)$ correct to six places of decimal.

11. Given: $\frac{dy}{dx} = y - x$ where $y(0) = 2$, using the Runge-Kutta's Fourth order method, find $y(0.1)$ and $y(0.2)$. Compare the approximate solution with its exact solution.

$$(e^{0.1} = 1.10517,\ e^{0.2} = 1.2214)$$

12. Apply the Second order Runge-Kutta's method, to find an approximate value of y at $x = 0.2$ taking $h = 0.1$, given that y satisfies the differential equation and the initial condition

$$y' = x + y$$
$$y(0) = 1$$

13. Using Runge-Kutta's method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$. Initial conditions are at $x = 0$, $y = 1$ and $y' = 0$ use four decimal places for computations.

14. $\frac{dy}{dx} = xy;\ y = 2$, when $x = 1$, use Runge-Kutta fourth order rule to find y at $x = 1.4$ taking the step length $h = 0.2$.

15. Apply modified Euler's method to determine $y(0.1)$, given that $\frac{dy}{dx} = x^2 + y$ when $y(0) = -1$.

16. Solve the Initial Value problem $\frac{dy}{dx} = \frac{1}{x + y}$, $y(0) = 1$ using Runge-Kutta method of fourth order to evaluate $y(0.5)$ in a single step.

17. Using Runge-Kutta method, find $y(0.2)$ given that

$$\frac{dy}{dx} = 1 + y^2,\ y(0) = 0.$$

18. Use Euler's method and an interval of $h = 0.1$ to find a solution of the differential equation $\frac{dy}{dx} = x^2 + y^2$ with Boundary conditions $y = 1$ for $x = 0$ in $0 \le x \le 1$.

19. Using Runge-Kutta method, find $y(2)$, given that $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$.

20. Drive Euler's method to solve an initial value problem of first order. How can this method be improved to obtain Runge-Kutta's explicit method of fourth order for the same problem? Explain.

21. When the initial value problem is $y' = 1 + y^2$, $y(0) = 0$. Find $y(0.6)$ by Runge-Kutta fourth order method (Taking $h = 0.2$).

22. Find the value of y (1.1) using Runge-Kutta method of fourth order, given that

$$\frac{dy}{dx} = y^2 + 3x,\ y(1) = 1.2. \qquad \text{[take } h = 0.1\text{]}$$

Answers-3

1.	$y(0.2) = 0.9802,\ y'(0.2) = 0.196$
2.	0.9477
3.	$y^{(3)} = \frac{x^5}{15} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \frac{22}{15}$
4.	$y_{1.4} = 2.99485866$
5.	0.76972 .
6.	$y = 4.2214$
7.	$y = 1.2104$
10.	$u(0.4) = 0.8602978$
12.	$x_2 = 0.2,\ y_2 = 1.2428$
13.	$\begin{cases} z(0.2) = 0.20133347 \\ y(0.2) = 0.9799326 \end{cases}$
14.	$y = 2.99486$
15.	$y = -1.105$
16.	$y = 1.357$
17.	$y = 0.20271$
18.	$y = 1, 1.1, 1.222, 1.375, 1.573$
21.	$y = 0.6828$
22.	$y = 1.727$

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